Single Variable Calculus

Early Transcendentals

MTH 251

This text was initially written by David Guichard. The single variable material in chapters 1–9 is a modification and expansion of notes written by Neal Koblitz at the University of Washington, who generously gave permission to use, modify, and distribute his work. New material has been added, and old material has been modified, so some portions now bear little resemblance to the original.

The book includes some exercises and examples from *Elementary Calculus: An Approach Using Infinitesimals*, by H. Jerome Keisler, available at http://www.math.wisc.edu/~keisler/calc.html under a Creative Commons license. In addition, the chapter on differential equations (in the multivariable version) and the section on numerical integration are largely derived from the corresponding portions of Keisler's book. Albert Schueller, Barry Balof, and Mike Wills have contributed additional material.

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Introduction

The emphasis in this course is on problems—doing calculations and story problems. To master problem solving one needs a tremendous amount of practice doing problems. The more problems you do the better you will be at doing them, as patterns will start to emerge in both the problems and in successful approaches to them. You will learn fastest and best if you devote some time to doing problems every day.

Typically the most difficult problems are story problems, since they require some effort before you can begin calculating. Here are some pointers for doing story problems:

1. Carefully read each problem twice before writing anything.
2. Assign letters to quantities that are described only in words; draw a diagram if appropriate.
3. Decide which letters are constants and which are variables. A letter stands for a constant if its value remains the same throughout the problem.
4. Using mathematical notation, write down what you know and then write down what you want to find.
5. Decide what category of problem it is (this might be obvious if the problem comes at the end of a particular chapter, but will not necessarily be so obvious if it comes on an exam covering several chapters).
6. Double check each step as you go along; don't wait until the end to check your work.
7. Use common sense; if an answer is out of the range of practical possibilities, then check your work to see where you went wrong.

Suggestions for Using This Text

1. Read the example problems carefully, filling in any steps that are left out (ask someone for help if you can't follow the solution to a worked example).
2. Later use the worked examples to study by covering the solutions, and seeing if you can solve the problems on your own.
3. Most exercises have answers in Appendix A; the availability of an answer is marked by "⇒" at the end of the exercise. In the pdf version of the full text, clicking on the arrow will take you to the answer. The answers should be used only as a final check on your work, not as a crutch. Keep in mind that sometimes an answer could be expressed in various ways that are algebraically equivalent, so don't assume that your answer is wrong just because it doesn't have exactly the same form as the answer in the back.
4. A few figures in the pdf and print versions of the book are marked with "(AP)" at the end of the caption. Clicking on this should open a related interactive applet or Sage worksheet in your web browser. Occasionally another link will do the same thing, like this example. (Note to users of a printed text: the words "this example" in the pdf file are blue, and are a link to a Sage worksheet.)

2

Instantaneous Rate of Change: The Derivative

2.1 The slope of a function

Suppose that y is a function of x, say $y = f(x)$. It is often necessary to know how sensitive the value of y is to small changes in x.

EXAMPLE 2.1.1 Take, for example, $y = f(x) = \sqrt{625 - x^2}$ (the upper semicircle of radius 25 centered at the origin). When $x = 7$, we find that $y = \sqrt{625 - 49} = 24$. Suppose we want to know how much y changes when x increases a little, say to 7.1 or 7.01.

In the case of a straight line $y = mx + b$, the slope $m = \Delta y/\Delta x$ measures the change in y per unit change in x. This can be interpreted as a measure of "sensitivity"; for example, if $y = 100x + 5$, a small change in x corresponds to a change one hundred times as large in y, so y is quite sensitive to changes in x.

Let us look at the same ratio $\Delta y/\Delta x$ for our function $y = f(x) = \sqrt{625 - x^2}$ when x changes from 7 to 7.1. Here $\Delta x = 7.1 - 7 = 0.1$ is the change in x, and

$$\begin{aligned} \Delta y &= f(x + \Delta x) - f(x) = f(7.1) - f(7) \\ &= \sqrt{625 - 7.1^2} - \sqrt{625 - 7^2} \approx 23.9706 - 24 = -0.0294. \end{aligned}$$

Thus, $\Delta y/\Delta x \approx -0.0294/0.1 = -0.294$. This means that y changes by less than one third the change in x, so apparently y is not very sensitive to changes in x at $x = 7$. We say "apparently" here because we don't really know what happens between 7 and 7.1. Perhaps y changes dramatically as x runs through the values from 7 to 7.1, but at 7.1 y just happens to be close to its value at 7. This is not in fact the case for this particular function, but we don't yet know why. □

One way to interpret the above calculation is by reference to a line. We have computed the slope of the line through $(7, 24)$ and $(7.1, 23.9706)$, called a **chord** of the circle. In general, if we draw the chord from the point $(7, 24)$ to a nearby point on the semicircle $(7 + \Delta x,\ f(7 + \Delta x))$, the slope of this chord is the so-called **difference quotient**

$$\text{slope of chord} = \frac{f(7 + \Delta x) - f(7)}{\Delta x} = \frac{\sqrt{625 - (7 + \Delta x)^2} - 24}{\Delta x}.$$

For example, if x changes only from 7 to 7.01, then the difference quotient (slope of the chord) is approximately equal to $(23.997081 - 24)/0.01 = -0.2919$. This is slightly less steep than the chord from $(7, 24)$ to $(7.1, 23.9706)$.

As the second value $7 + \Delta x$ moves in towards 7, the chord joining $(7, f(7))$ to $(7 + \Delta x, f(7 + \Delta x))$ shifts slightly. As indicated in figure 2.1.1, as Δx gets smaller and smaller, the chord joining $(7, 24)$ to $(7 + \Delta x, f(7 + \Delta x))$ gets closer and closer to the **tangent line** to the circle at the point $(7, 24)$. (Recall that the tangent line is the line that just grazes the circle at that point, i.e., it doesn't meet the circle at any second point.) Thus, as Δx gets smaller and smaller, the slope $\Delta y/\Delta x$ of the chord gets closer and closer to the slope of the tangent line. This is actually quite difficult to see when Δx is small, because of the scale of the graph. The values of Δx used for the figure are 1, 5, 10 and 15, not really very small values. The tangent line is the one that is uppermost at the right hand endpoint.

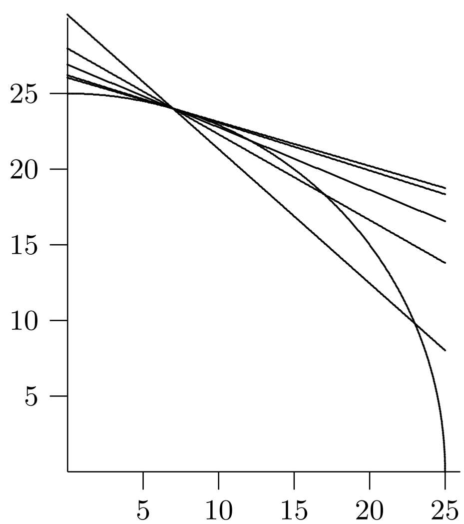


Figure 2.1.1 Chords approximating the tangent line. (AP)

So far we have found the slopes of two chords that should be close to the slope of the tangent line, but what is the slope of the tangent line exactly? Since the tangent line touches the circle at just one point, we will never be able to calculate its slope directly, using two "known" points on the line. What we need is a way to capture what happens to the slopes of the chords as they get "closer and closer" to the tangent line.

Instead of looking at more particular values of Δx, let's see what happens if we do some algebra with the difference quotient using just Δx. The slope of a chord from $(7, 24)$ to a nearby point is given by

$$\begin{aligned}
\frac{\sqrt{625-(7+\Delta x)^2}-24}{\Delta x} &= \frac{\sqrt{625-(7+\Delta x)^2}-24}{\Delta x}\frac{\sqrt{625-(7+\Delta x)^2}+24}{\sqrt{625-(7+\Delta x)^2}+24} \\
&= \frac{625-(7+\Delta x)^2-24^2}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{49-49-14\Delta x-\Delta x^2}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{\Delta x(-14-\Delta x)}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{-14-\Delta x}{\sqrt{625-(7+\Delta x)^2}+24}
\end{aligned}$$

Now, can we tell by looking at this last formula what happens when Δx gets very close to zero? The numerator clearly gets very close to -14 while the denominator gets very close to $\sqrt{625-7^2}+24=48$. Is the fraction therefore very close to $-14/48=-7/24\cong-0.29167$? It certainly seems reasonable, and in fact it is true: as Δx gets closer and closer to zero, the difference quotient does in fact get closer and closer to $-7/24$, and so the slope of the tangent line is exactly $-7/24$.

What about the slope of the tangent line at $x=12$? Well, 12 can't be all that different from 7; we just have to redo the calculation with 12 instead of 7. This won't be hard, but it will be a bit tedious. What if we try to do all the algebra without using a specific value for x? Let's copy from above, replacing 7 by x. We'll have to do a bit more than that—for example, the "24" in the calculation came from $\sqrt{625-7^2}$, so we'll need to fix that too.

$$\begin{aligned}
&\frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x} = \\
&= \frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x}\frac{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}} \\
&= \frac{625-(x+\Delta x)^2-625+x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{625-x^2-2x\Delta x-\Delta x^2-625+x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{\Delta x(-2x-\Delta x)}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{-2x-\Delta x}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}
\end{aligned}$$

Now what happens when Δx is very close to zero? Again it seems apparent that the quotient will be very close to

$$\frac{-2x}{\sqrt{625-x^2}+\sqrt{625-x^2}} = \frac{-2x}{2\sqrt{625-x^2}} = \frac{-x}{\sqrt{625-x^2}}.$$

Replacing x by 7 gives $-7/24$, as before, and now we can easily do the computation for 12 or any other value of x between -25 and 25.

So now we have a single, simple formula, $-x/\sqrt{625-x^2}$, that tells us the slope of the tangent line for any value of x. This slope, in turn, tells us how sensitive the value of y is to changes in the value of x.

What do we call such a formula? That is, a formula with one variable, so that substituting an "input" value for the variable produces a new "output" value? This is a function. Starting with one function, $\sqrt{625-x^2}$, we have derived, by means of some slightly nasty algebra, a new function, $-x/\sqrt{625-x^2}$, that gives us important information about the original function. This new function in fact is called the **derivative** of the original function. If the original is referred to as f or y then the derivative is often written f' or y' and pronounced "f prime" or "y prime", so in this case we might write $f'(x) = -x/\sqrt{625-x^2}$. At a particular point, say $x = 7$, we say that $f'(7) = -7/24$ or "f prime of 7 is $-7/24$" or "the derivative of f at 7 is $-7/24$."

To summarize, we compute the derivative of $f(x)$ by forming the difference quotient

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}, \tag{2.1.1}$$

which is the slope of a line, then we figure out what happens when Δx gets very close to 0.

We should note that in the particular case of a circle, there's a simple way to find the derivative. Since the tangent to a circle at a point is perpendicular to the radius drawn to the point of contact, its slope is the negative reciprocal of the slope of the radius. The radius joining $(0, 0)$ to $(7, 24)$ has slope $24/7$. Hence, the tangent line has slope $-7/24$. In general, a radius to the point $(x, \sqrt{625 - x^2})$ has slope $\sqrt{625 - x^2}/x$, so the slope of the tangent line is $-x/\sqrt{625 - x^2}$, as before. It is **NOT** always true that a tangent line is perpendicular to a line from the origin—don't use this shortcut in any other circumstance.

As above, and as you might expect, for different values of x we generally get different values of the derivative $f'(x)$. Could it be that the derivative always has the same value? This would mean that the slope of f, or the slope of its tangent line, is the same everywhere. One curve that always has the same slope is a line; it seems odd to talk about the tangent line to a line, but if it makes sense at all the tangent line must be the line itself. It is not hard to see that the derivative of $f(x) = mx + b$ is $f'(x) = m$; see exercise 6.

Exercises 2.1.

1. Draw the graph of the function $y = f(x) = \sqrt{169 - x^2}$ between $x = 0$ and $x = 13$. Find the slope $\Delta y/\Delta x$ of the chord between the points of the circle lying over (a) $x = 12$ and $x = 13$, (b) $x = 12$ and $x = 12.1$, (c) $x = 12$ and $x = 12.01$, (d) $x = 12$ and $x = 12.001$. Now use the geometry of tangent lines on a circle to find (e) the exact value of the derivative $f'(12)$. Your answers to (a)–(d) should be getting closer and closer to your answer to (e). ⇒

2. Use geometry to find the derivative $f'(x)$ of the function $f(x) = \sqrt{625 - x^2}$ in the text for each of the following x: (a) 20, (b) 24, (c) -7, (d) -15. Draw a graph of the upper semicircle, and draw the tangent line at each of these four points. ⇒

3. Draw the graph of the function $y = f(x) = 1/x$ between $x = 1/2$ and $x = 4$. Find the slope of the chord between (a) $x = 3$ and $x = 3.1$, (b) $x = 3$ and $x = 3.01$, (c) $x = 3$ and $x = 3.001$. Now use algebra to find a simple formula for the slope of the chord between $(3, f(3))$ and $(3 + \Delta x, f(3 + \Delta x))$. Determine what happens when Δx approaches 0. In your graph of $y = 1/x$, draw the straight line through the point $(3, 1/3)$ whose slope is this limiting value of the difference quotient as Δx approaches 0. ⇒

4. Find an algebraic expression for the difference quotient $\big(f(1+\Delta x) - f(1)\big)/\Delta x$ when $f(x) = x^2 - (1/x)$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(1)$. ⇒

5. Draw the graph of $y = f(x) = x^3$ between $x = 0$ and $x = 1.5$. Find the slope of the chord between (a) $x = 1$ and $x = 1.1$, (b) $x = 1$ and $x = 1.001$, (c) $x = 1$ and $x = 1.00001$. Then use algebra to find a simple formula for the slope of the chord between 1 and $1 + \Delta x$. (Use the expansion $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.) Determine what happens as Δx approaches 0, and in your graph of $y = x^3$ draw the straight line through the point $(1, 1)$ whose slope is equal to the value you just found. ⇒

6. Find an algebraic expression for the difference quotient $(f(x+\Delta x) - f(x))/\Delta x$ when $f(x) = mx + b$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(x)$. ⇒

7. Sketch the unit circle. Discuss the behavior of the slope of the tangent line at various angles around the circle. Which trigonometric function gives the slope of the tangent line at an angle θ? Why? Hint: think in terms of ratios of sides of triangles.

8. Sketch the parabola $y = x^2$. For what values of x on the parabola is the slope of the tangent line positive? Negative? What do you notice about the graph at the point(s) where the sign of the slope changes from positive to negative and vice versa?

2.2 An example

We started the last section by saying, "It is often necessary to know how sensitive the value of y is to small changes in x." We have seen one purely mathematical example of this: finding the "steepness" of a curve at a point is precisely this problem. Here is a more applied example.

With careful measurement it might be possible to discover that a dropped ball has height $h(t) = h_0 - kt^2$, t seconds after it is released. (Here h_0 is the initial height of the ball, when $t = 0$, and k is some number determined by the experiment.) A natural question is then, "How fast is the ball going at time t?" We can certainly get a pretty good idea with a little simple arithmetic. To make the calculation more concrete, let's say $h_0 = 100$ meters and $k = 4.9$ and suppose we're interested in the speed at $t = 2$. We know that when $t = 2$ the height is $100 - 4 \cdot 4.9 = 80.4$. A second later, at $t = 3$, the height is $100 - 9 \cdot 4.9 = 55.9$, so in that second the ball has traveled $80.4 - 55.9 = 24.5$ meters. This means that the *average* speed during that time was 24.5 meters per second. So we might guess that 24.5 meters per second is not a terrible estimate of the speed at $t = 2$. But certainly we can do better. At $t = 2.5$ the height is $100 - 4.9(2.5)^2 = 69.375$. During the half second from $t = 2$ to $t = 2.5$ the ball dropped $80.4 - 69.375 = 11.025$ meters, at an average speed of $11.025/(1/2) = 22.05$ meters per second; this should be a better estimate of the speed at $t = 2$. So it's clear now how to get better and better approximations: compute average speeds over shorter and shorter time intervals. Between $t = 2$ and $t = 2.01$, for example, the ball drops 0.19649 meters in one hundredth of a second, at an average speed of 19.649 meters per second.

We can't do this forever, and we still might reasonably ask what the actual speed precisely at $t = 2$ is. If Δt is some tiny amount of time, what we want to know is what happens to the average speed $(h(2) - h(2 + \Delta t))/\Delta t$ as Δt gets smaller and smaller. Doing a bit of algebra:

$$\begin{aligned}\frac{h(2)-h(2+\Delta t)}{\Delta t} &= \frac{80.4-(100-4.9(2+\Delta t)^2)}{\Delta t}\\ &= \frac{80.4-100+19.6+19.6\Delta t+4.9\Delta t^2}{\Delta t}\\ &= \frac{19.6\Delta t+4.9\Delta t^2}{\Delta t}\\ &= 19.6+4.9\Delta t\end{aligned}$$

When Δt is very small, this is very close to 19.6, and indeed it seems clear that as Δt goes to zero, the average speed goes to 19.6, so the exact speed at $t = 2$ is 19.6 meters per second. This calculation should look very familiar. In the language of the previous section, we might have started with $f(x) = 100 - 4.9x^2$ and asked for the slope of the tangent line at $x = 2$. We would have answered that question by computing

$$\frac{f(2+\Delta x)-f(2)}{\Delta x} = \frac{-19.6\Delta x-4.9\Delta x^2}{\Delta x} = -19.6-4.9\Delta x$$

The algebra is the same, except that following the pattern of the previous section the subtraction would be reversed, and we would say that the slope of the tangent line is -19.6. Indeed, in hindsight, perhaps we should have subtracted the other way even for the dropping ball. At $t = 2$ the height is 80.4; one second later the height is 55.9. The usual way to compute a “distance traveled” is to subtract the earlier position from the later one, or $55.9 - 80.4 = -24.5$. This tells us that the distance traveled is 24.5 meters, and the negative sign tells us that the height went down during the second. If we continue the original calculation we then get -19.6 meters per second as the exact speed at $t = 2$. If we interpret the negative sign as meaning that the motion is downward, which seems reasonable, then in fact this is the same answer as before, but with even more information, since the numerical answer contains the direction of motion as well as the speed. Thus, the speed of the ball is the value of the derivative of a certain function, namely, of the function that gives the position of the ball. (More properly, this is the *velocity* of the ball; velocity is signed speed, that is, speed with a direction indicated by the sign.)

The upshot is that this problem, finding the speed of the ball, is *exactly* the same problem mathematically as finding the slope of a curve. This may already be enough evidence to convince you that whenever some quantity is changing (the height of a curve or the height of a ball or the size of the economy or the distance of a space probe from earth or the population of the world) the rate at which the quantity is changing can, in principle, be computed in exactly the same way, by finding a derivative.

Exercises 2.2.

1. An object is traveling in a straight line so that its position (that is, distance from some fixed point) is given by this table:

time (seconds)	0	1	2	3
distance (meters)	0	10	25	60

Find the average speed of the object during the following time intervals: $[0, 1]$, $[0, 2]$, $[0, 3]$, $[1, 2]$, $[1, 3]$, $[2, 3]$. If you had to guess the speed at $t = 2$ just on the basis of these, what would you guess? ⇒

2. Let $y = f(t) = t^2$, where t is the time in seconds and y is the distance in meters that an object falls on a certain airless planet. Draw a graph of this function between $t = 0$ and $t = 3$. Make a table of the average speed of the falling object between (a) 2 sec and 3 sec, (b) 2 sec and 2.1 sec, (c) 2 sec and 2.01 sec, (d) 2 sec and 2.001 sec. Then use algebra to find a simple formula for the average speed between time 2 and time $2 + \Delta t$. (If you substitute $\Delta t = 1$, 0.1, 0.01, 0.001 in this formula you should again get the answers to parts (a)–(d).) Next, in your formula for average speed (which should be in simplified form) determine what happens as Δt approaches zero. This is the instantaneous speed. Finally, in your graph of $y = t^2$ draw the straight line through the point $(2, 4)$ whose slope is the instantaneous velocity you just computed; it should of course be the tangent line. ⇒

3. If an object is dropped from an 80-meter high window, its height y above the ground at time t seconds is given by the formula $y = f(t) = 80 - 4.9t^2$. (Here we are neglecting air resistance; the graph of this function was shown in figure 1.0.1.) Find the average velocity of the falling object between (a) 1 sec and 1.1 sec, (b) 1 sec and 1.01 sec, (c) 1 sec and 1.001 sec. Now use algebra to find a simple formula for the average velocity of the falling object between 1 sec and $1 + \Delta t$ sec. Determine what happens to this average velocity as Δt approaches 0. That is the instantaneous velocity at time $t = 1$ second (it will be negative, because the object is falling). ⇒

2.3 Limits

In the previous two sections we computed some quantities of interest (slope, velocity) by seeing that some expression "goes to" or "approaches" or "gets really close to" a particular value. In the examples we saw, this idea may have been clear enough, but it is too fuzzy to rely on in more difficult circumstances. In this section we will see how to make the idea more precise.

There is an important feature of the examples we have seen. Consider again the formula

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x}.$$

We wanted to know what happens to this fraction as "Δx goes to zero." Because we were able to simplify the fraction, it was easy to see the answer, but it was not quite as simple as "substituting zero for Δx," as that would give

$$\frac{-19.6 \cdot 0 - 4.9 \cdot 0}{0},$$

which is meaningless. The quantity we are really interested in does not make sense "at zero," and this is why the answer to the original problem (finding a velocity or a slope) was not immediately obvious. In other words, we are generally going to want to figure out what a quantity "approaches" in situations where we can't merely plug in a value. If you would like to think about a hard example (which we will analyze later) consider what happens to $(\sin x)/x$ as x approaches zero.

EXAMPLE 2.3.1 Does $\sqrt{x}$ approach 1.41 as x approaches 2? In this case it is possible to compute the actual value $\sqrt{2}$ to a high precision to answer the question. But since in general we won't be able to do that, let's not. We might start by computing $\sqrt{x}$ for values of x close to 2, as we did in the previous sections. Here are some values: $\sqrt{2.05} = 1.431782106$, $\sqrt{2.04} = 1.428285686$, $\sqrt{2.03} = 1.424780685$, $\sqrt{2.02} = 1.421267040$, $\sqrt{2.01} = 1.417744688$, $\sqrt{2.005} = 1.415980226$, $\sqrt{2.004} = 1.415627070$, $\sqrt{2.003} = 1.415273825$, $\sqrt{2.002} = 1.414920492$, $\sqrt{2.001} = 1.414567072$. So it looks at least possible that indeed these values "approach" 1.41—already $\sqrt{2.001}$ is quite close. If we continue this process, however, at some point we will appear to "stall." In fact, $\sqrt{2} = 1.414213562\ldots$, so we will never even get as far as 1.4142, no matter how long we continue the sequence. □

So in a fuzzy, everyday sort of sense, it is true that $\sqrt{x}$ "gets close to" 1.41, but it does not "approach" 1.41 in the sense we want. To compute an exact slope or an exact velocity, what we want to know is that a given quantity becomes "arbitrarily close" to a fixed value, meaning that the first quantity can be made "as close as we like" to the fixed value. Consider again the quantities

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x} = -19.6 - 4.9\Delta x.$$

These two quantities are equal as long as Δx is not zero; if Δx is zero, the left hand quantity is meaningless, while the right hand one is -19.6. Can we say more than we did before about why the right hand side "approaches" -19.6, in the desired sense? Can we really make it "as close as we want" to -19.6? Let's try a test case. Can we make $-19.6 - 4.9\Delta x$ within one millionth (0.000001) of -19.6? The values within a millionth of -19.6 are those in the interval $(-19.600001, -19.599999)$. As Δx approaches zero, does $-19.6 - 4.9\Delta x$ eventually reside inside this interval? If Δx is positive, this would require that $-19.6 - 4.9\Delta x > -19.600001$. This is something we can manipulate with a little algebra:

$$
\begin{aligned}
-19.6 - 4.9\Delta x &> -19.600001 \\
-4.9\Delta x &> -0.000001 \\
\Delta x &< -0.000001/-4.9 \\
\Delta x &< 0.0000002040816327\ldots
\end{aligned}
$$

Thus, we can say with certainty that if Δx is positive and less than 0.0000002, then $\Delta x < 0.0000002040816327\ldots$ and so $-19.6 - 4.9\Delta x > -19.600001$. We could do a similar calculation if Δx is negative.

So now we know that we can make $-19.6 - 4.9\Delta x$ within one millionth of -19.6. But can we make it "as close as we want"? In this case, it is quite simple to see that the answer is yes, by modifying the calculation we've just done. It may be helpful to think of this as a game. I claim that I can make $-19.6 - 4.9\Delta x$ as close as you desire to -19.6 by making Δx "close enough" to zero. So the game is: you give me a number, like 10^{-6}, and I have to come up with a number representing how close Δx must be to zero to guarantee that $-19.6 - 4.9\Delta x$ is at least as close to -19.6 as you have requested.

Now if we actually play this game, I could redo the calculation above for each new number you provide. What I'd like to do is somehow see that I will always succeed, and even more, I'd like to have a simple strategy so that I don't have to do all that algebra every time. A strategy in this case would be a formula that gives me a correct answer no matter what you specify. So suppose the number you give me is ϵ. How close does Δx have to be to zero to guarantee that $-19.6 - 4.9\Delta x$ is in $(-19.6 - \epsilon, -19.6 + \epsilon)$? If Δx is positive, we need:

$$
\begin{aligned}
-19.6 - 4.9\Delta x &> -19.6 - \epsilon \\
-4.9\Delta x &> -\epsilon \\
\Delta x &< -\epsilon/-4.9 \\
\Delta x &< \epsilon/4.9
\end{aligned}
$$

So if I pick any number δ that is less than $\epsilon/4.9$, the algebra tells me that whenever $\Delta x < \delta$ then $\Delta x < \epsilon/4.9$ and so $-19.6 - 4.9\Delta x$ is within ϵ of -19.6. (This is exactly what I did in the example: I picked $\delta = 0.0000002 < 0.0000002040816327\ldots$.) A similar calculation again works for negative Δx. The important fact is that this is now a completely general result—it shows that I can always win, no matter what "move" you make.

Now we can codify this by giving a precise definition to replace the fuzzy, "gets closer and closer" language we have used so far. Henceforward, we will say something like "the limit of $(-19.6\Delta x - 4.9\Delta x^2)/\Delta x$ as Δx goes to zero is -19.6," and abbreviate this mouthful as

$$\lim_{\Delta x \to 0} \frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x} = -19.6.$$

Here is the actual, official definition of "limit".

DEFINITION 2.3.2 Limit Suppose f is a function. We say that $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x - a| < \delta$, $|f(x) - L| < \epsilon$. □

The ϵ and δ here play exactly the role they did in the preceding discussion. The definition says, in a very precise way, that $f(x)$ can be made as close as desired to L (that's the $|f(x) - L| < \epsilon$ part) by making x close enough to a (the $0 < |x - a| < \delta$ part). Note that we specifically make no mention of what must happen if $x = a$, that is, if $|x - a| = 0$. This is because in the cases we are most interested in, substituting a for x doesn't even make sense.

Make sure you are not confused by the names of important quantities. The generic definition talks about $f(x)$, but the function and the variable might have other names. In the discussion above, the function we analyzed was

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x}.$$

and the variable of the limit was not x but Δx. The x was the variable of the original function; when we were trying to compute a slope or a velocity, x was essentially a fixed quantity, telling us at what point we wanted the slope. (In the velocity problem, it was literally a fixed quantity, as we focused on the time 2.) The quantity a of the definition in all the examples was zero: we were always interested in what happened as Δx became very close to zero.

Armed with a precise definition, we can now prove that certain quantities behave in a particular way. The bad news is that even proofs for simple quantities can be quite tedious and complicated; the good news is that we rarely need to do such proofs, because most expressions act the way you would expect, and this can be proved once and for all.

EXAMPLE 2.3.3 Let's show carefully that $\lim_{x \to 2} x + 4 = 6$. This is not something we "need" to prove, since it is "obviously" true. But if we couldn't prove it using our official definition there would be something very wrong with the definition.

As is often the case in mathematical proofs, it helps to work backwards. We want to end up showing that under certain circumstances $x + 4$ is close to 6; precisely, we want to show that $|x + 4 - 6| < \epsilon$, or $|x - 2| < \epsilon$. Under what circumstances? We want this to be true whenever $0 < |x - 2| < \delta$. So the question becomes: can we choose a value for δ that guarantees that $0 < |x-2| < \delta$ implies $|x-2| < \epsilon$? Of course: no matter what ϵ is, $\delta = \epsilon$ works. □

So it turns out to be very easy to prove something "obvious," which is nice. It doesn't take long before things get trickier, however.

EXAMPLE 2.3.4 It seems clear that $\lim_{x\to 2} x^2 = 4$. Let's try to prove it. We will want to be able to show that $|x^2-4| < \epsilon$ whenever $0 < |x-2| < \delta$, by choosing δ carefully. Is there any connection between $|x-2|$ and $|x^2-4|$? Yes, and it's not hard to spot, but it is not so simple as the previous example. We can write $|x^2-4| = |(x+2)(x-2)|$. Now when $|x-2|$ is small, part of $|(x+2)(x-2)|$ is small, namely $(x-2)$. What about $(x+2)$? If x is close to 2, $(x+2)$ certainly can't be too big, but we need to somehow be precise about it. Let's recall the "game" version of what is going on here. You get to pick an ϵ and I have to pick a δ that makes things work out. Presumably it is the really tiny values of ϵ I need to worry about, but I have to be prepared for anything, even an apparently "bad" move like $\epsilon = 1000$. I expect that ϵ is going to be small, and that the corresponding δ will be small, certainly less than 1. If $\delta \le 1$ then $|x+2| < 5$ when $|x-2| < \delta$ (because if x is within 1 of 2, then x is between 1 and 3 and $x+2$ is between 3 and 5). So then I'd be trying to show that $|(x+2)(x-2)| < 5|x-2| < \epsilon$. So now how can I pick δ so that $|x-2| < \delta$ implies $5|x-2| < \epsilon$? This is easy: use $\delta = \epsilon/5$, so $5|x-2| < 5(\epsilon/5) = \epsilon$. But what if the ϵ you choose is not small? If you choose $\epsilon = 1000$, should I pick $\delta = 200$? No, to keep things "sane" I will never pick a δ bigger than 1. Here's the final "game strategy:" When you pick a value for ϵ I will pick $\delta = \epsilon/5$ or $\delta = 1$, whichever is smaller. Now when $|x-2| < \delta$, I know both that $|x+2| < 5$ and that $|x-2| < \epsilon/5$. Thus $|(x+2)(x-2)| < 5(\epsilon/5) = \epsilon$.

This has been a long discussion, but most of it was explanation and scratch work. If this were written down as a proof, it would be quite short, like this:

Proof that $\lim_{x\to 2} x^2 = 4$. Given any ϵ, pick $\delta = \epsilon/5$ or $\delta = 1$, whichever is smaller. Then when $|x-2| < \delta$, $|x+2| < 5$ and $|x-2| < \epsilon/5$. Hence $|x^2-4| = |(x+2)(x-2)| < 5(\epsilon/5) = \epsilon$. □

It probably seems obvious that $\lim_{x\to 2} x^2 = 4$, and it is worth examining more closely why it seems obvious. If we write $x^2 = x \cdot x$, and ask what happens when x approaches 2, we might say something like, "Well, the first x approaches 2, and the second x approaches 2, so the product must approach $2 \cdot 2$." In fact this is pretty much right on the money, except for that word "must." Is it really true that if x approaches a and y approaches b then xy approaches ab? It is, but it is not really obvious, since x and y might be quite complicated. The good news is that we can see that this is true once and for all, and then we don't have to worry about it ever again. When we say that x might be "complicated" we really mean that in practice it might be a function. Here is then what we want to know:

THEOREM 2.3.5 Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then $\lim_{x\to a} f(x)g(x) = LM$.

Proof. We have to use the official definition of limit to make sense of this. So given any ϵ we need to find a δ so that $0 < |x-a| < \delta$ implies $|f(x)g(x) - LM| < \epsilon$. What do we have to work with? We know that we can make $f(x)$ close to L and $g(x)$ close to M, and we have to somehow connect these facts to make $f(x)g(x)$ close to LM.

We use, as is so often the case, a little algebraic trick:

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\
&= |f(x)(g(x) - M) + (f(x) - L)M| \\
&\le |f(x)(g(x) - M)| + |(f(x) - L)M| \\
&= |f(x)||g(x) - M| + |f(x) - L||M|.
\end{aligned}$$

This is all straightforward except perhaps for the "$\le$". That is an example of the **triangle inequality** , which says that if a and b are any real numbers then $|a+b| \le |a| + |b|$. If you look at a few examples, using positive and negative numbers in various combinations for a and b, you should quickly understand why this is true; we will not prove it formally.

Since $\lim_{x\to a} f(x) = L$, there is a value δ_1 so that $0 < |x-a| < \delta_1$ implies $|f(x) - L| < |\epsilon/(2M)|$, This means that $0 < |x-a| < \delta_1$ implies $|f(x) - L||M| < \epsilon/2$. You can see where this is going: if we can make $|f(x)||g(x) - M| < \epsilon/2$ also, then we'll be done.

We can make $|g(x) - M|$ smaller than any fixed number by making x close enough to a; unfortunately, $\epsilon/(2f(x))$ is not a fixed number, since x is a variable. Here we need another little trick, just like the one we used in analyzing x^2. We can find a δ_2 so that $|x-a| < \delta_2$ implies that $|f(x) - L| < 1$, meaning that $L - 1 < f(x) < L + 1$. This means that $|f(x)| < N$, where N is either $|L-1|$ or $|L+1|$, depending on whether L is negative or positive. The important point is that N doesn't depend on x. Finally, we know that there is a δ_3 so that $0 < |x-a| < \delta_3$ implies $|g(x) - M| < \epsilon/(2N)$. Now we're ready to put everything together. Let δ be the smallest of δ_1, δ_2, and δ_3. Then $|x-a| < \delta$ implies that $|f(x) - L| < |\epsilon/(2M)|$, $|f(x)| < N$, and $|g(x) - M| < \epsilon/(2N)$. Then

$$\begin{aligned}
|f(x)g(x) - LM| &\le |f(x)||g(x) - M| + |f(x) - L||M| \\
&< N\frac{\epsilon}{2N} + \left|\frac{\epsilon}{2M}\right||M| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

This is just what we needed, so by the official definition, $\lim_{x\to a} f(x)g(x) = LM$. ∎

A handful of such theorems give us the tools to compute many limits without explicitly working with the definition of limit.

THEOREM 2.3.6 Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ and k is some constant. Then

$$\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = kL$$
$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L+M$$
$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L-M$$
$$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) = LM$$
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}, \text{ if } M \text{ is not } 0$$

■

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the "innermost bits" and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim_{x\to a} x = a$.

EXAMPLE 2.3.7 Compute $\lim_{x\to 1}\frac{x^2-3x+5}{x-2}$. If we apply the theorem in all its gory detail, we get

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2-3x+5}{x-2} &= \frac{\lim_{x\to 1}(x^2-3x+5)}{\lim_{x\to 1}(x-2)}\\
&= \frac{(\lim_{x\to 1} x^2) - (\lim_{x\to 1} 3x) + (\lim_{x\to 1} 5)}{(\lim_{x\to 1} x) - (\lim_{x\to 1} 2)}\\
&= \frac{(\lim_{x\to 1} x)^2 - 3(\lim_{x\to 1} x) + 5}{(\lim_{x\to 1} x) - 2}\\
&= \frac{1^2 - 3\cdot 1 + 5}{1-2}\\
&= \frac{1-3+5}{-1} = -3
\end{aligned}$$

□

It is worth commenting on the trivial limit $\lim_{x\to 1} 5$. From one point of view this might seem meaningless, as the number 5 can't "approach" any value, since it is simply a fixed number. But 5 can, and should, be interpreted here as the function that has value 5 everywhere, $f(x) = 5$, with graph a horizontal line. From this point of view it makes sense to ask what happens to the height of the function as x approaches 1.

Of course, as we've already seen, we're primarily interested in limits that aren't so easy, namely, limits in which a denominator approaches zero. There are a handful of algebraic tricks that work on many of these limits.

EXAMPLE 2.3.8 Compute $\lim_{x\to 1} \frac{x^2 + 2x - 3}{x - 1}$. We can't simply plug in $x = 1$ because that makes the denominator zero. However:

$$\begin{aligned}\lim_{x\to 1} \frac{x^2 + 2x - 3}{x - 1} &= \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1} \\ &= \lim_{x\to 1}(x+3) = 4\end{aligned}$$

□

While theorem 2.3.6 is very helpful, we need a bit more to work easily with limits. Since the theorem applies when some limits are already known, we need to know the behavior of some functions that cannot themselves be constructed from the simple arithmetic operations of the theorem, such as $\sqrt{x}$. Also, there is one other extraordinarily useful way to put functions together: composition. If $f(x)$ and $g(x)$ are functions, we can form two functions by composition: $f(g(x))$ and $g(f(x))$. For example, if $f(x) = \sqrt{x}$ and $g(x) = x^2 + 5$, then $f(g(x)) = \sqrt{x^2 + 5}$ and $g(f(x)) = (\sqrt{x})^2 + 5 = x + 5$. Here is a companion to theorem 2.3.6 for composition:

THEOREM 2.3.9 Suppose that $\lim_{x\to a} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$. Then

$$\lim_{x\to a} f(g(x)) = f(L).$$

■

Note the special form of the condition on f: it is not enough to know that $\lim_{x\to L} f(x) = M$, though it is a bit tricky to see why. Many of the most familiar functions do have this property, and this theorem can therefore be applied. For example:

THEOREM 2.3.10 Suppose that n is a positive integer. Then

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a},$$

provided that a is positive if n is even. ■

This theorem is not too difficult to prove from the definition of limit.

Another of the most common algebraic tricks was used in section 2.1. Here's another example:

EXAMPLE 2.3.11 Compute $\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}$.

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} &= \lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} \\
&= \lim_{x\to -1} \frac{x+5-4}{(x+1)(\sqrt{x+5}+2)} \\
&= \lim_{x\to -1} \frac{x+1}{(x+1)(\sqrt{x+5}+2)} \\
&= \lim_{x\to -1} \frac{1}{\sqrt{x+5}+2} = \frac{1}{4}
\end{aligned}$$

At the very last step we have used theorems 2.3.9 and 2.3.10. □

Occasionally we will need a slightly modified version of the limit definition. Consider the function $f(x) = \sqrt{1-x^2}$, the upper half of the unit circle. What can we say about $\lim_{x\to 1} f(x)$? It is apparent from the graph of this familiar function that as x gets close to 1 from the left, the value of $f(x)$ gets close to zero. It does not even make sense to ask what happens as x approaches 1 from the right, since $f(x)$ is not defined there. The definition of the limit, however, demands that $f(1+\Delta x)$ be close to $f(1)$ whether Δx is positive or negative. Sometimes the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to be able to talk about this situation, we introduce the concept of **one sided limit**:

DEFINITION 2.3.12 **One-sided limit** Suppose that $f(x)$ is a function. We say that $\lim_{x\to a^-} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < a - x < \delta$, $|f(x) - L| < \epsilon$. We say that $\lim_{x\to a^+} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < x - a < \delta$, $|f(x) - L| < \epsilon$. □

Usually $\lim_{x\to a^-} f(x)$ is read "the limit of $f(x)$ from the left" and $\lim_{x\to a^+} f(x)$ is read "the limit of $f(x)$ from the right".

EXAMPLE 2.3.13 Discuss $\lim_{x\to 0} \frac{x}{|x|}$, $\lim_{x\to 0^-} \frac{x}{|x|}$, and $\lim_{x\to 0^+} \frac{x}{|x|}$.

The function $f(x) = x/|x|$ is undefined at 0; when $x > 0$, $|x| = x$ and so $f(x) = 1$; when $x < 0$, $|x| = -x$ and $f(x) = -1$. Thus $\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} -1 = -1$ while $\lim_{x\to 0^+} \frac{x}{|x|} =$

$\lim_{x\to 0^+} 1 = 1$. The limit of $f(x)$ must be equal to both the left and right limits; since they are different, the limit $\lim_{x\to 0} \frac{x}{|x|}$ does not exist. □

Exercises 2.3.

Compute the limits. If a limit does not exist, explain why.

1. $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$ ⇒

2. $\lim_{x\to 1} \frac{x^2+x-12}{x-3}$ ⇒

3. $\lim_{x\to -4} \frac{x^2+x-12}{x-3}$ ⇒

4. $\lim_{x\to 2} \frac{x^2+x-12}{x-2}$ ⇒

5. $\lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1}$ ⇒

6. $\lim_{x\to 0^+} \sqrt{\frac{1}{x}+2} - \sqrt{\frac{1}{x}}.$ ⇒

7. $\lim_{x\to 2} 3$ ⇒

8. $\lim_{x\to 4} 3x^3 - 5x$ ⇒

9. $\lim_{x\to 0} \frac{4x-5x^2}{x-1}$ ⇒

10. $\lim_{x\to 1} \frac{x^2-1}{x-1}$ ⇒

11. $\lim_{x\to 0^+} \frac{\sqrt{2-x^2}}{x}$ ⇒

12. $\lim_{x\to 0^+} \frac{\sqrt{2-x^2}}{x+1}$ ⇒

13. $\lim_{x\to a} \frac{x^3-a^3}{x-a}$ ⇒

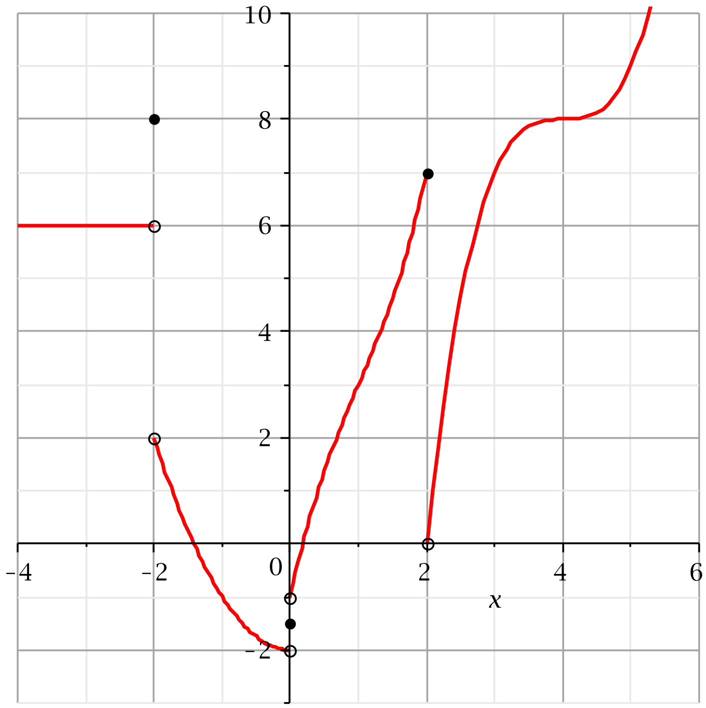
14. $\lim_{x\to 2} (x^2+4)^3$ ⇒

15. $\lim_{x\to 1} \begin{cases} x-5 & x\neq 1, \\ 7 & x = 1. \end{cases}$ ⇒

16. $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$ (Hint: Use the fact that $|\sin a| < 1$ for any real number a. You should probably use the definition of a limit here.) ⇒

17. Give an ϵ–δ proof, similar to example 2.3.3, of the fact that $\lim_{x\to 4}(2x-5) = 3$.

18. Evaluate the expressions by reference to this graph:



(a) $\lim_{x\to 4} f(x)$ (b) $\lim_{x\to -3} f(x)$ (c) $\lim_{x\to 0} f(x)$

(d) $\lim_{x\to 0^-} f(x)$ (e) $\lim_{x\to 0^+} f(x)$ (f) $f(-2)$

(g) $\lim_{x\to 2^-} f(x)$ (h) $\lim_{x\to -2^-} f(x)$ (i) $\lim_{x\to 0} f(x+1)$

(j) $f(0)$ (k) $\lim_{x\to 1^-} f(x-4)$ (l) $\lim_{x\to 0^+} f(x-2)$

$\Rightarrow$

19. Use a calculator to estimate $\lim_{x\to 0} \frac{\sin x}{x}$.

20. Use a calculator to estimate $\lim_{x\to 0} \frac{\tan(3x)}{\tan(5x)}$.

2.4 The Derivative Function

We have seen how to create, or derive, a new function $f'(x)$ from a function $f(x)$, summarized in the paragraph containing equation 2.1.1. Now that we have the concept of limits, we can make this more precise.

DEFINITION 2.4.1 The derivative of a function f, denoted f', is

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

□

We know that f' carries important information about the original function f. In one example we saw that $f'(x)$ tells us how steep the graph of $f(x)$ is; in another we saw that $f'(x)$ tells us the velocity of an object if $f(x)$ tells us the position of the object at time x. As we said earlier, this same mathematical idea is useful whenever $f(x)$ represents some changing quantity and we want to know something about how it changes, or roughly, the "rate" at which it changes. Most functions encountered in practice are built up from a small collection of "primitive" functions in a few simple ways, for example, by adding or multiplying functions together to get new, more complicated functions. To make good use of the information provided by $f'(x)$ we need to be able to compute it for a variety of such functions.

We will begin to use different notations for the derivative of a function. While initially confusing, each is often useful so it is worth maintaining multiple versions of the same thing.

Consider again the function $f(x) = \sqrt{625 - x^2}$. We have computed the derivative $f'(x) = -x/\sqrt{625 - x^2}$, and have already noted that if we use the alternate notation $y = \sqrt{625 - x^2}$ then we might write $y' = -x/\sqrt{625 - x^2}$. Another notation is quite different, and in time it will become clear why it is often a useful one. Recall that to compute the the derivative of f we computed

$$\lim_{\Delta x \to 0} \frac{\sqrt{625 - (7 + \Delta x)^2} - 24}{\Delta x}.$$

The denominator here measures a distance in the x direction, sometimes called the "run", and the numerator measures a distance in the y direction, sometimes called the "rise," and "rise over run" is the slope of a line. Recall that sometimes such a numerator is abbreviated Δy, exchanging brevity for a more detailed expression. So in general, a derivative is given by

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

To recall the form of the limit, we sometimes say instead that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

In other words, dy/dx is another notation for the derivative, and it reminds us that it is related to an actual slope between two points. This notation is called **Leibniz notation**, after Gottfried Leibniz, who developed the fundamentals of calculus independently, at about the same time that Isaac Newton did. Again, since we often use f and $f(x)$ to mean the original function, we sometimes use df/dx and $df(x)/dx$ to refer to the derivative. If the function $f(x)$ is written out in full we often write the last of these something like this

$$f'(x) = \frac{d}{dx}\sqrt{625 - x^2}$$

with the function written to the side, instead of trying to fit it into the numerator.

EXAMPLE 2.4.2 Find the derivative of $y = f(t) = t^2$.

We compute

$$\begin{aligned} y' = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} &= \lim_{\Delta t \to 0} \frac{(t + \Delta t)^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{t^2 + 2t\Delta t + \Delta t^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{2t\Delta t + \Delta t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} 2t + \Delta t = 2t. \end{aligned}$$

Remember that Δt is a single quantity, not a "Δ" times a "t", and so Δt^2 is $(\Delta t)^2$ not $\Delta(t^2)$. □

EXAMPLE 2.4.3 Find the derivative of $y = f(x) = 1/x$.

The computation:

$$\begin{aligned} y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{x}{x(x+\Delta x)} - \frac{x+\Delta x}{x(x+\Delta x)}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x - x - \Delta x}{x(x + \Delta x)\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-\Delta x}{x(x + \Delta x)\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-1}{x(x + \Delta x)} = \frac{-1}{x^2} \end{aligned}$$

□

Note. If you happen to know some "derivative formulas" from an earlier course, for the time being you should pretend that you do not know them. In examples like the ones above and the exercises below, you are required to know how to find the derivative formula starting from basic principles. We will later develop some formulas so that we do not always need to do such computations, but we will continue to need to know how to do the more involved computations.

Sometimes one encounters a point in the domain of a function $y = f(x)$ where there is *no derivative*, because there is no tangent line. In order for the notion of the tangent line at a point to make sense, the curve must be "smooth" at that point. This means that if you imagine a particle traveling at some steady speed along the curve, then the particle does not experience an abrupt change of direction. There are two types of situations you should be aware of—corners and cusps—where there's a sudden change of direction and hence no derivative.

EXAMPLE 2.4.4 Discuss the derivative of the absolute value function $y = f(x) = |x|$.

If x is positive, then this is the function $y = x$, whose derivative is the constant 1. (Recall that when $y = f(x) = mx + b$, the derivative is the slope m.) If x is negative, then we're dealing with the function $y = -x$, whose derivative is the constant -1. If $x = 0$, then the function has a corner, i.e., there is no tangent line. A tangent line would have to point in the direction of the curve—but there are *two* directions of the curve that come together at the origin. We can summarize this as

$$y' = \begin{cases} 1 & \text{if } x > 0; \\ -1 & \text{if } x < 0; \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

□

EXAMPLE 2.4.5

Discuss the derivative of the function $y = x^{2/3}$, shown in figure 2.4.1. We will later see how to compute this derivative; for now we use the fact that $y' = (2/3)x^{-1/3}$. Visually this looks much like the absolute value function, but it technically has a cusp, not a corner. The absolute value function has no tangent line at 0 because there are (at least) two obvious contenders—the tangent line of the left side of the curve and the tangent line of the right side. The function $y = x^{2/3}$ does not have a tangent line at 0, but unlike the absolute value function it can be said to have a single direction: as we approach 0 from either side the tangent line becomes closer and closer to a vertical line; the curve is vertical at 0. But as before, if you imagine traveling along the curve, an abrupt change in direction is required at 0: a full 180 degree turn. □

In practice we won't worry much about the distinction between these examples; in both cases the function has a "sharp point" where there is no tangent line and no derivative.

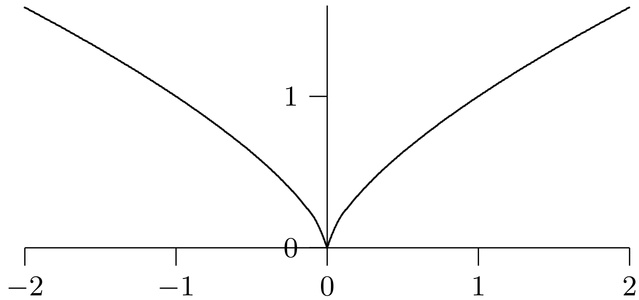
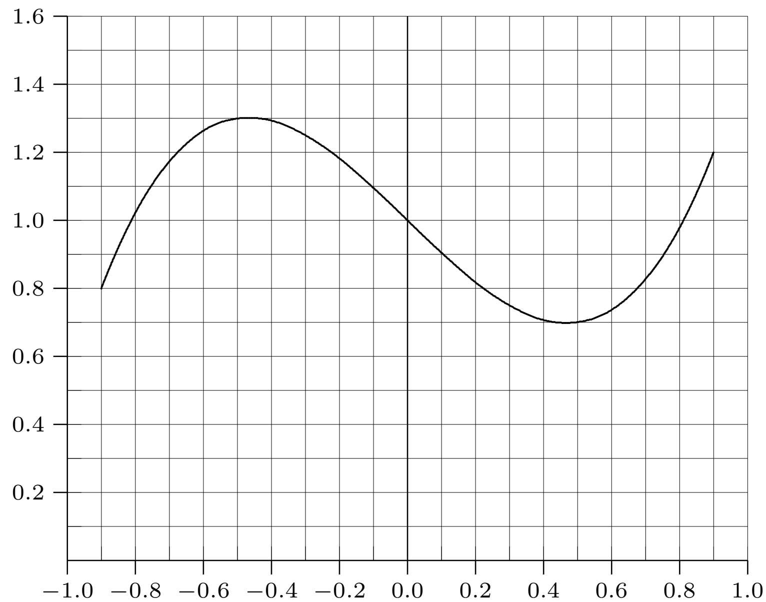


Figure 2.4.1 A cusp on $x^{2/3}$.

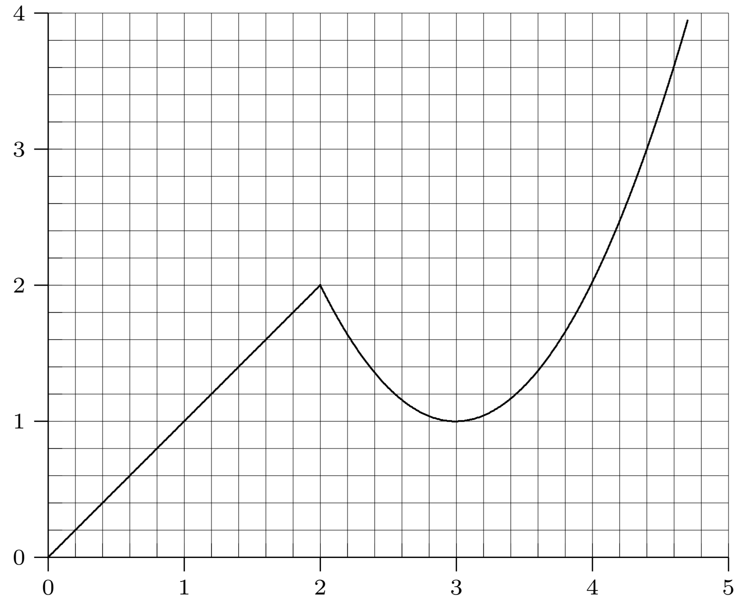
Exercises 2.4.

1. Find the derivative of $y = f(x) = \sqrt{169 - x^2}$. ⇒
2. Find the derivative of $y = f(t) = 80 - 4.9t^2$. ⇒
3. Find the derivative of $y = f(x) = x^2 - (1/x)$. ⇒
4. Find the derivative of $y = f(x) = ax^2 + bx + c$ (where a, b, and c are constants). ⇒
5. Find the derivative of $y = f(x) = x^3$. ⇒
6. Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero. Make sure you indicate any places where the derivative does not exist.



7. Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero.

Make sure you indicate any places where the derivative does not exist.



8. Find the derivative of $y = f(x) = 2/\sqrt{2x+1}$ ⇒

9. Find the derivative of $y = g(t) = (2t-1)/(t+2)$ ⇒

10. Find an equation for the tangent line to the graph of $f(x) = 5 - x - 3x^2$ at the point $x = 2$ ⇒

11. Find a value for a so that the graph of $f(x) = x^2 + ax - 3$ has a horizontal tangent line at $x = 4$. ⇒

2.5 Adjectives For Functions

As we have defined it in Section 1.3, a function is a very general object. At this point, it is useful to introduce a collection of adjectives to describe certain kinds of functions; these adjectives name useful properties that functions may have. Consider the graphs of the functions in Figure 2.5.1. It would clearly be useful to have words to help us describe the distinct features of each of them. We will point out and define a few adjectives (there are many more) for the functions pictured here. For the sake of the discussion, we will assume that the graphs do not exhibit any unusual behavior off-stage (i.e., outside the view of the graphs).

Functions. Each graph in Figure 2.5.1 certainly represents a function—since each passes the *vertical line test*. In other words, as you sweep a vertical line across the graph of each function, the line never intersects the graph more than once. If it did, then the graph would not represent a function.

Bounded. The graph in (c) appears to approach zero as x goes to both positive and negative infinity. It also never exceeds the value 1 or drops below the value 0. Because the

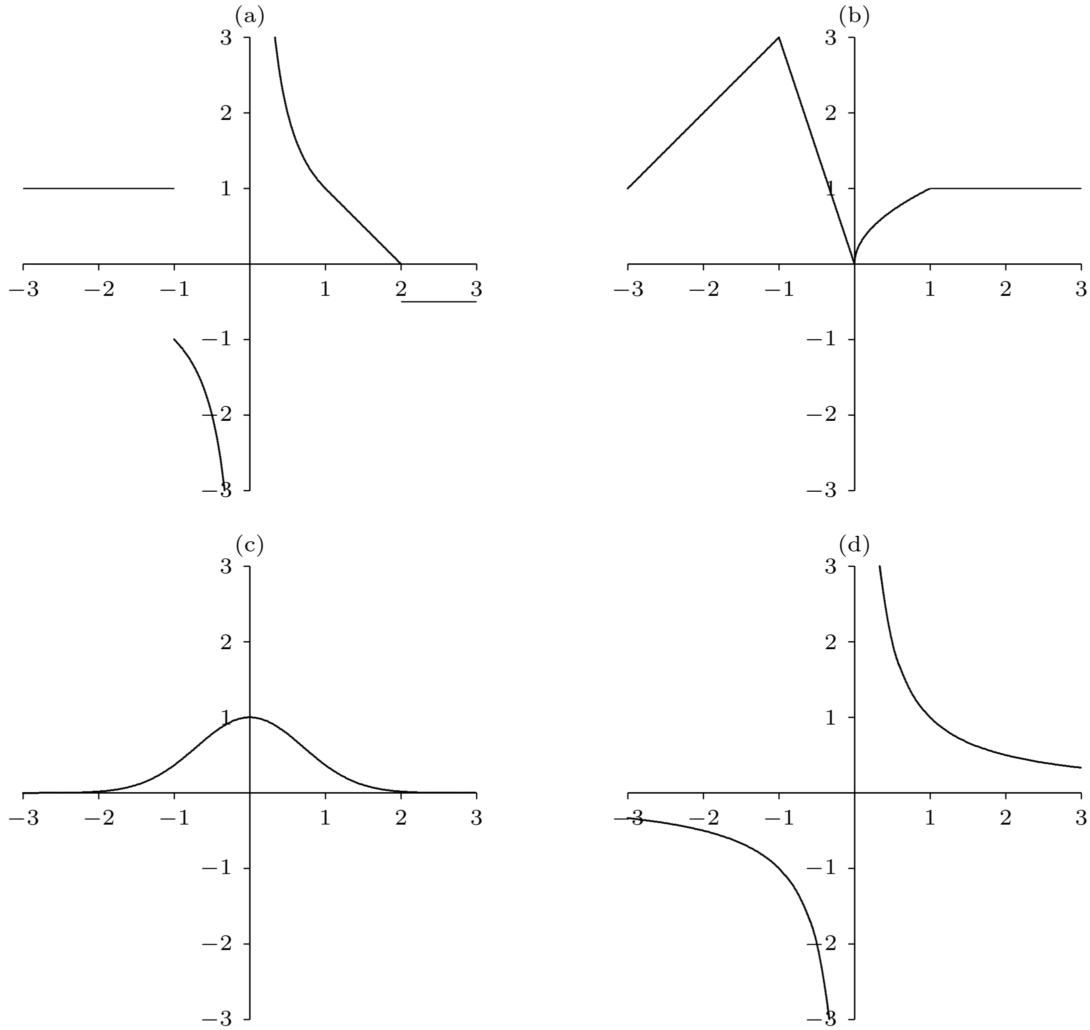


Figure 2.5.1 Function Types: (a) a discontinuous function, (b) a continuous function, (c) a bounded, differentiable function, (d) an unbounded, differentiable function

graph never increases or decreases without bound, we say that the function represented by the graph in (c) is a **bounded** function.

DEFINITION 2.5.1 Bounded A function f is bounded if there is a number M such that $|f(x)| < M$ for every x in the domain of f. □

For the function in (c), one such choice for M would be 10. However, the smallest (optimal) choice would be $M = 1$. In either case, simply finding an M is enough to establish boundedness. No such M exists for the hyperbola in (d) and hence we can say that it is **unbounded**.

Continuity. The graphs shown in (b) and (c) both represent **continuous** functions. Geometrically, this is because there are no jumps in the graphs. That is, if you pick a point on the graph and approach it from the left and right, the values of the function approach the value of the function at that point. For example, we can see that this is not true for function values near $x = -1$ on the graph in (a) which is not continuous at that location.

DEFINITION 2.5.2 Continuous at a Point A function f is continuous at a point a if $\lim_{x \to a} f(x) = f(a)$. □

DEFINITION 2.5.3 Continuous A function f is continuous if it is continuous at every point in its domain. □

Strangely, we can also say that (d) is continuous even though there is a vertical asymptote. A careful reading of the definition of continuous reveals the phrase "*at every point in its domain.*" Because the location of the asymptote, $x = 0$, is not in the domain of the function, and because the rest of the function is *well-behaved*, we can say that (d) is continuous.

Differentiability. Now that we have introduced the derivative of a function at a point, we can begin to use the adjective **differentiable**. We can see that the tangent line is well-defined at every point on the graph in (c). Therefore, we can say that (c) is a differentiable function.

DEFINITION 2.5.4 Differentiable at a Point A function f is differentiable at point a if $f'(a)$ exists. □

DEFINITION 2.5.5 Differentiable A function f is differentiable if is differentiable at every point (excluding endpoints and isolated points in the domain of f) in the domain of f. □

Take note that, for technical reasons not discussed here, both of these definitions exclude endpoints and isolated points in the domain from consideration.

We now have a collection of adjectives to describe the very rich and complex set of objects known as functions.

We close with a useful theorem about continuous functions:

THEOREM 2.5.6 Intermediate Value Theorem If f is continuous on the interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$. ■

This is most frequently used when $d = 0$.

EXAMPLE 2.5.7 Explain why the function $f = x^3 + 3x^2 + x - 2$ has a root between 0 and 1.

By theorem 2.3.6, f is continuous. Since $f(0) = -2$ and $f(1) = 3$, and 0 is between -2 and 3, there is a $c \in [0, 1]$ such that $f(c) = 0$. □

This example also points the way to a simple method for approximating roots.

EXAMPLE 2.5.8 Approximate the root of the previous example to one decimal place.

If we compute $f(0.1)$, $f(0.2)$, and so on, we find that $f(0.6) < 0$ and $f(0.7) > 0$, so by the Intermediate Value Theorem, f has a root between 0.6 and 0.7. Repeating the process with $f(0.61)$, $f(0.62)$, and so on, we find that $f(0.61) < 0$ and $f(0.62) > 0$, so f has a root between 0.61 and 0.62, and the root is 0.6 rounded to one decimal place. □

Exercises 2.5.

1. Along the lines of Figure 2.5.1, for each part below sketch the graph of a function that is:

  a. bounded, but not continuous.

  b. differentiable and unbounded.

  c. continuous at $x = 0$, not continuous at $x = 1$, and bounded.

  d. differentiable everywhere except at $x = -1$, continuous, and unbounded.

2. Is $f(x) = \sin(x)$ a bounded function? If so, find the smallest M.

3. Is $s(t) = 1/(1 + t^2)$ a bounded function? If so, find the smallest M.

4. Is $v(u) = 2 \ln |u|$ a bounded function? If so, find the smallest M.

5. Consider the function

$$h(x) = \begin{cases} 2x - 3, & \text{if } x < 1 \\ 0, & \text{if } x \geq 1. \end{cases}$$

Show that it is continuous at the point $x = 0$. Is h a continuous function?

6. Approximate a root of $f = x^3 - 4x^2 + 2x + 2$ to one decimal place. ⇒

7. Approximate a root of $f = x^4 + x^3 - 5x + 1$ to one decimal place.

3

Rules for Finding Derivatives

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to compute the derivative of almost any function we are likely to encounter. Many functions involve quantities raised to a constant power, such as polynomials and more complicated combinations like $y = (\sin x)^4$. So we start by examining powers of a single variable; this gives us a building block for more complicated examples.

3.1 The Power Rule

We start with the derivative of a power function, $f(x) = x^n$. Here n is a number of any kind: integer, rational, positive, negative, even irrational, as in x^π. We have already computed some simple examples, so the formula should not be a complete surprise:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

It is not easy to show this is true for any n. We will do some of the easier cases now, and discuss the rest later.

The easiest, and most common, is the case that n is a positive integer. To compute the derivative we need to compute the following limit:

$$\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}.$$

For a specific, fairly small value of n, we could do this by straightforward algebra.

EXAMPLE 3.1.1 Find the derivative of $f(x) = x^3$.

$$\begin{aligned}
\frac{d}{dx}x^3 &= \lim_{\Delta x\to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}.\\
&= \lim_{\Delta x\to 0} \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - x^3}{\Delta x}.\\
&= \lim_{\Delta x\to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x}.\\
&= \lim_{\Delta x\to 0} 3x^2 + 3x\Delta x + \Delta x^2 = 3x^2.
\end{aligned}$$

□

The general case is really not much harder as long as we don't try to do too much. The key is understanding what happens when $(x + \Delta x)^n$ is multiplied out:

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + a_2 x^{n-2}\Delta x^2 + \cdots + +a_{n-1}x\Delta x^{n-1} + \Delta x^n.$$

We know that multiplying out will give a large number of terms all of the form $x^i \Delta x^j$, and in fact that $i + j = n$ in every term. One way to see this is to understand that one method for multiplying out $(x + \Delta x)^n$ is the following: In every $(x + \Delta x)$ factor, pick either the x or the Δx, then multiply the n choices together; do this in all possible ways. For example, for $(x + \Delta x)^3$, there are eight possible ways to do this:

$$\begin{aligned}
(x+\Delta x)(x+\Delta x)(x+\Delta x) &= xxx + xx\Delta x + x\Delta x x + x\Delta x\Delta x\\
&\quad + \Delta x xx + \Delta x x\Delta x + \Delta x\Delta x x + \Delta x\Delta x\Delta x\\
&= x^3 + x^2\Delta x + x^2\Delta x + x\Delta x^2\\
&\quad + x^2\Delta x + x\Delta x^2 + x\Delta x^2 + \Delta x^3\\
&= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3
\end{aligned}$$

No matter what n is, there are n ways to pick Δx in one factor and x in the remaining $n-1$ factors; this means one term is $nx^{n-1}\Delta x$. The other coefficients are somewhat harder to understand, but we don't really need them, so in the formula above they have simply been called a_2, a_3, and so on. We know that every one of these terms contains Δx to at least the power 2. Now let's look at the limit:

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{\Delta x\to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}\\
&= \lim_{\Delta x\to 0} \frac{x^n + nx^{n-1}\Delta x + a_2x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n - x^n}{\Delta x}\\
&= \lim_{\Delta x\to 0} \frac{nx^{n-1}\Delta x + a_2x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n}{\Delta x}\\
&= \lim_{\Delta x\to 0} nx^{n-1} + a_2x^{n-2}\Delta x + \cdots + a_{n-1}x\Delta x^{n-2} + \Delta x^{n-1} = nx^{n-1}.
\end{aligned}$$

Now without much trouble we can verify the formula for negative integers. First let's look at an example:

EXAMPLE 3.1.2 Find the derivative of $y = x^{-3}$. Using the formula, $y' = -3x^{-3-1} = -3x^{-4}$. □

Here is the general computation. Suppose n is a negative integer; the algebra is easier to follow if we use $n = -m$ in the computation, where m is a positive integer.

$$\begin{aligned}
\frac{d}{dx}x^n = \frac{d}{dx}x^{-m} &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^{-m} - x^{-m}}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\frac{1}{(x+\Delta x)^m} - \frac{1}{x^m}}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{x^m - (x+\Delta x)^m}{(x+\Delta x)^m x^m \Delta x}\\
&= \lim_{\Delta x\to 0}\frac{x^m - (x^m + mx^{m-1}\Delta x + a_2 x^{m-2}\Delta x^2 + \cdots + a_{m-1}x\Delta x^{m-1} + \Delta x^m)}{(x+\Delta x)^m x^m \Delta x}\\
&= \lim_{\Delta x\to 0}\frac{-mx^{m-1} - a_2 x^{m-2}\Delta x - \cdots - a_{m-1}x\Delta x^{m-2} - \Delta x^{m-1})}{(x+\Delta x)^m x^m}\\
&= \frac{-mx^{m-1}}{x^m x^m} = \frac{-mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = nx^{-m-1} = nx^{n-1}.
\end{aligned}$$

We will later see why the other cases of the power rule work, but from now on we will use the power rule whenever n is any real number. Let's note here a simple case in which the power rule applies, or almost applies, but is not really needed. Suppose that $f(x) = 1$; remember that this "1" is a function, not "merely" a number, and that $f(x) = 1$ has a graph that is a horizontal line, with slope zero everywhere. So we know that $f'(x) = 0$. We might also write $f(x) = x^0$, though there is some question about just what this means at $x = 0$. If we apply the power rule, we get $f'(x) = 0x^{-1} = 0/x = 0$, again noting that there is a problem at $x = 0$. So the power rule "works" in this case, but it's really best to just remember that the derivative of any constant function is zero.

Exercises 3.1.

Find the derivatives of the given functions.

1. x^{100} ⇒

2. x^{-100} ⇒

3. $\dfrac{1}{x^5}$ ⇒

4. x^{π} ⇒

5. $x^{3/4}$ ⇒

6. $x^{-9/7}$ ⇒

3.2 Linearity of the Derivative

An operation is linear if it behaves "nicely" with respect to multiplication by a constant and addition. The name comes from the equation of a line through the origin, $f(x) = mx$, and the following two properties of this equation. First, $f(cx) = m(cx) = c(mx) = cf(x)$, so the constant c can be "moved outside" or "moved through" the function f. Second, $f(x+y) = m(x+y) = mx + my = f(x) + f(y)$, so the addition symbol likewise can be moved through the function.

The corresponding properties for the derivative are:

$$(cf(x))' = \frac{d}{dx}cf(x) = c\frac{d}{dx}f(x) = cf'(x),$$

and

$$(f(x)+g(x))' = \frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x).$$

It is easy to see, or at least to believe, that these are true by thinking of the distance/speed interpretation of derivatives. If one object is at position $f(t)$ at time t, we know its speed is given by $f'(t)$. Suppose another object is at position $5f(t)$ at time t, namely, that it is always 5 times as far along the route as the first object. Then it "must" be going 5 times as fast at all times.

The second rule is somewhat more complicated, but here is one way to picture it. Suppose a flat bed railroad car is at position $f(t)$ at time t, so the car is traveling at a speed of $f'(t)$ (to be specific, let's say that $f(t)$ gives the position on the track of the rear end of the car). Suppose that an ant is crawling from the back of the car to the front so that its position *on the car* is $g(t)$ and its speed *relative to the car* is $g'(t)$. Then in reality, at time t, the ant is at position $f(t) + g(t)$ along the track, and its speed is "obviously" $f'(t) + g'(t)$.

We don't want to rely on some more-or-less obvious physical interpretation to determine what is true mathematically, so let's see how to verify these rules by computation.

We'll do one and leave the other for the exercises.

$$\begin{aligned}
\frac{d}{dx}(f(x)+g(x)) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-(f(x)+g(x))}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}\right)\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\\
&= f'(x)+g'(x)
\end{aligned}$$

This is sometimes called the **sum rule** for derivatives.

EXAMPLE 3.2.1 Find the derivative of $f(x)=x^5+5x^2$. We have to invoke linearity twice here:

$$f'(x)=\frac{d}{dx}(x^5+5x^2)=\frac{d}{dx}x^5+\frac{d}{dx}(5x^2)=5x^4+5\frac{d}{dx}(x^2)=5x^4+5\cdot 2x^1=5x^4+10x.$$

□

Because it is so easy with a little practice, we can usually combine all uses of linearity into a single step. The following example shows an acceptably detailed computation.

EXAMPLE 3.2.2 Find the derivative of $f(x)=3/x^4-2x^2+6x-7$.

$$f'(x)=\frac{d}{dx}\left(\frac{3}{x^4}-2x^2+6x-7\right)=\frac{d}{dx}(3x^{-4}-2x^2+6x-7)=-12x^{-5}-4x+6.$$

□

Exercises 3.2.

Find the derivatives of the functions in 1–6.

1. $5x^3+12x^2-15$ ⇒

2. $-4x^5+3x^2-5/x^2$ ⇒

3. $5(-3x^2+5x+1)$ ⇒

4. $f(x)+g(x)$, where $f(x)=x^2-3x+2$ and $g(x)=2x^3-5x$ ⇒

5. $(x+1)(x^2+2x-3)$ ⇒

6. $\sqrt{625-x^2}+3x^3+12$ (See section 2.1.) ⇒

7. Find an equation for the tangent line to $f(x)=x^3/4-1/x$ at $x=-2$. ⇒

8. Find an equation for the tangent line to $f(x) = 3x^2 - \pi^3$ at $x = 4$. ⇒

9. Suppose the position of an object at time t is given by $f(t) = -49t^2/10 + 5t + 10$. Find a function giving the speed of the object at time t. The acceleration of an object is the rate at which its speed is changing, which means it is given by the derivative of the speed function. Find the acceleration of the object at time t. ⇒

10. Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of f, cf, f', and $(cf)'$ on the same diagram.

11. The general polynomial P of degree n in the variable x has the form $P(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \ldots + a_n x^n$. What is the derivative (with respect to x) of P? ⇒

12. Find a cubic polynomial whose graph has horizontal tangents at $(-2, 5)$ and $(2, 3)$. ⇒

13. Prove that $\frac{d}{dx}(cf(x)) = cf'(x)$ using the definition of the derivative.

14. Suppose that f and g are differentiable at x. Show that $f - g$ is differentiable at x using the two linearity properties from this section.

3.3 The Product Rule

Consider the product of two simple functions, say $f(x) = (x^2 + 1)(x^3 - 3x)$. An obvious guess for the derivative of f is the product of the derivatives of the constituent functions: $(2x)(3x^2 - 3) = 6x^3 - 6x$. Is this correct? We can easily check, by rewriting f and doing the calculation in a way that is known to work. First, $f(x) = x^5 - 3x^3 + x^3 - 3x = x^5 - 2x^3 - 3x$, and then $f'(x) = 5x^4 - 6x^2 - 3$. Not even close! What went "wrong"? Well, nothing really, except the guess was wrong.

So the derivative of $f(x)g(x)$ is NOT as simple as $f'(x)g'(x)$. Surely there is some rule for such a situation? There is, and it is instructive to "discover" it by trying to do the general calculation even without knowing the answer in advance.

$$\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} g(x) \\
&= f(x)g'(x) + f'(x)g(x)
\end{aligned}$$

A couple of items here need discussion. First, we used a standard trick, "add and subtract the same thing", to transform what we had into a more useful form. After some rewriting, we realize that we have two limits that produce $f'(x)$ and $g'(x)$. Of course, $f'(x)$ and

$g'(x)$ must actually exist for this to make sense. We also replaced $\lim_{\Delta x\to 0} f(x+\Delta x)$ with $f(x)$—why is this justified?

What we really need to know here is that $\lim_{\Delta x\to 0} f(x+\Delta x) = f(x)$, or in the language of section 2.5, that f is continuous at x. We already know that $f'(x)$ exists (or the whole approach, writing the derivative of fg in terms of f' and g', doesn't make sense). This turns out to imply that f is continuous as well. Here's why:

$$\begin{aligned}\lim_{\Delta x\to 0} f(x+\Delta x) &= \lim_{\Delta x\to 0}(f(x+\Delta x) - f(x) + f(x))\\ &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}\Delta x + \lim_{\Delta x\to 0} f(x)\\ &= f'(x)\cdot 0 + f(x) = f(x)\end{aligned}$$

To summarize: the product rule says that

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x).$$

Returning to the example we started with, let $f(x) = (x^2+1)(x^3-3x)$. Then $f'(x) = (x^2+1)(3x^2-3) + (2x)(x^3-3x) = 3x^4 - 3x^2 + 3x^2 - 3 + 2x^4 - 6x^2 = 5x^4 - 6x^2 - 3$, as before. In this case it is probably simpler to multiply $f(x)$ out first, then compute the derivative; here's an example for which we really need the product rule.

EXAMPLE 3.3.1 Compute the derivative of $f(x) = x^2\sqrt{625-x^2}$. We have already computed $\frac{d}{dx}\sqrt{625-x^2} = \frac{-x}{\sqrt{625-x^2}}$. Now

$$f'(x) = x^2\frac{-x}{\sqrt{625-x^2}} + 2x\sqrt{625-x^2} = \frac{-x^3+2x(625-x^2)}{\sqrt{625-x^2}} = \frac{-3x^3+1250x}{\sqrt{625-x^2}}.$$

□

Exercises 3.3.

In 1–4, find the derivatives of the functions using the product rule.

1. $x^3(x^3-5x+10)$ ⇒

2. $(x^2+5x-3)(x^5-6x^3+3x^2-7x+1)$ ⇒

3. $\sqrt{x}\sqrt{625-x^2}$ ⇒

4. $\frac{\sqrt{625-x^2}}{x^{20}}$ ⇒

5. Use the product rule to compute the derivative of $f(x) = (2x-3)^2$. Sketch the function. Find an equation of the tangent line to the curve at $x = 2$. Sketch the tangent line at $x = 2$. ⇒

6. Suppose that f, g, and h are differentiable functions. Show that $(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

7. State and prove a rule to compute $(fghi)'(x)$, similar to the rule in the previous problem.

Product notation. Suppose $f_1, f_2, \ldots f_n$ are functions. The product of all these functions can be written

$$\prod_{k=1}^{n} f_k.$$

This is similar to the use of $\sum$ to denote a sum. For example,

$$\prod_{k=1}^{5} f_k = f_1 f_2 f_3 f_4 f_5$$

and

$$\prod_{k=1}^{n} k = 1 \cdot 2 \cdot \ldots \cdot n = n!.$$

We sometimes use somewhat more complicated conditions; for example

$$\prod_{k=1, k\neq j}^{n} f_k$$

denotes the product of f_1 through f_n except for f_j. For example,

$$\prod_{k=1, k\neq 4}^{5} x^k = x \cdot x^2 \cdot x^3 \cdot x^5 = x^{11}.$$

8. The **generalized product rule** says that if $f_1, f_2, \ldots, f_n$ are differentiable functions at x then

$$\frac{d}{dx} \prod_{k=1}^{n} f_k(x) = \sum_{j=1}^{n} \left(f_j'(x) \prod_{k=1, k\neq j}^{n} f_k(x) \right).$$

Verify that this is the same as your answer to the previous problem when $n = 4$, and write out what this says when $n = 5$.

3.4 The Quotient Rule

What is the derivative of $(x^2+1)/(x^3-3x)$? More generally, we'd like to have a formula to compute the derivative of $f(x)/g(x)$ if we already know $f'(x)$ and $g'(x)$. Instead of attacking this problem head-on, let's notice that we've already done part of the problem: $f(x)/g(x) = f(x) \cdot (1/g(x))$, that is, this is "really" a product, and we can compute the derivative if we know $f'(x)$ and $(1/g(x))'$. So really the only new bit of information we need is $(1/g(x))'$ in terms of $g'(x)$. As with the product rule, let's set this up and see how far we can get:

$$\begin{aligned}\frac{d}{dx}\frac{1}{g(x)} &= \lim_{\Delta x\to 0}\frac{\frac{1}{g(x+\Delta x)}-\frac{1}{g(x)}}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\frac{g(x)-g(x+\Delta x)}{g(x+\Delta x)g(x)}}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{g(x)-g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x}\\ &= \lim_{\Delta x\to 0} -\frac{g(x+\Delta x)-g(x)}{\Delta x}\frac{1}{g(x+\Delta x)g(x)}\\ &= -\frac{g'(x)}{g(x)^2}\end{aligned}$$

Now we can put this together with the product rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = f(x)\frac{-g'(x)}{g(x)^2}+f'(x)\frac{1}{g(x)} = \frac{-f(x)g'(x)+f'(x)g(x)}{g(x)^2} = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}.$$

EXAMPLE 3.4.1 Compute the derivative of $(x^2+1)/(x^3-3x)$.

$$\frac{d}{dx}\frac{x^2+1}{x^3-3x} = \frac{2x(x^3-3x)-(x^2+1)(3x^2-3)}{(x^3-3x)^2} = \frac{-x^4-6x^2+3}{(x^3-3x)^2}.$$

□

It is often possible to calculate derivatives in more than one way, as we have already seen. Since every quotient can be written as a product, it is always possible to use the product rule to compute the derivative, though it is not always simpler.

EXAMPLE 3.4.2 Find the derivative of $\sqrt{625-x^2}/\sqrt{x}$ in two ways: using the quotient rule, and using the product rule.

Quotient rule:

$$\frac{d}{dx}\frac{\sqrt{625-x^2}}{\sqrt{x}} = \frac{\sqrt{x}(-x/\sqrt{625-x^2})-\sqrt{625-x^2}\cdot 1/(2\sqrt{x})}{x}.$$

Note that we have used $\sqrt{x}=x^{1/2}$ to compute the derivative of $\sqrt{x}$ by the power rule.

Product rule:

$$\frac{d}{dx}\sqrt{625-x^2}x^{-1/2} = \sqrt{625-x^2}\frac{-1}{2}x^{-3/2}+\frac{-x}{\sqrt{625-x^2}}x^{-1/2}.$$

With a bit of algebra, both of these simplify to

$$-\frac{x^2+625}{2\sqrt{625-x^2}x^{3/2}}.$$

□

Occasionally you will need to compute the derivative of a quotient with a constant numerator, like $10/x^2$. Of course you can use the quotient rule, but it is usually not the easiest method. If we do use it here, we get

$$\frac{d}{dx}\frac{10}{x^2} = \frac{x^2 \cdot 0 - 10 \cdot 2x}{x^4} = \frac{-20}{x^3},$$

since the derivative of 10 is 0. But it is simpler to do this:

$$\frac{d}{dx}\frac{10}{x^2} = \frac{d}{dx}10x^{-2} = -20x^{-3}.$$

Admittedly, x^2 is a particularly simple denominator, but we will see that a similar calculation is usually possible. Another approach is to remember that

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2},$$

but this requires extra memorization. Using this formula,

$$\frac{d}{dx}\frac{10}{x^2} = 10\frac{-2x}{x^4}.$$

Note that we first use linearity of the derivative to pull the 10 out in front.

Exercises 3.4.

Find the derivatives of the functions in 1–4 using the quotient rule.

1. $\dfrac{x^3}{x^3 - 5x + 10}$ ⇒

2. $\dfrac{x^2 + 5x - 3}{x^5 - 6x^3 + 3x^2 - 7x + 1}$ ⇒

3. $\dfrac{\sqrt{x}}{\sqrt{625 - x^2}}$ ⇒

4. $\dfrac{\sqrt{625 - x^2}}{x^{20}}$ ⇒

5. Find an equation for the tangent line to $f(x) = (x^2 - 4)/(5 - x)$ at $x = 3$. ⇒

6. Find an equation for the tangent line to $f(x) = (x - 2)/(x^3 + 4x - 1)$ at $x = 1$. ⇒

7. Let P be a polynomial of degree n and let Q be a polynomial of degree m (with Q not the zero polynomial). Using sigma notation we can write

$$P = \sum_{k=0}^{n} a_k x^k, \qquad Q = \sum_{k=0}^{m} b_k x^k.$$

Use sigma notation to write the derivative of the **rational function** P/Q.

8. The curve $y = 1/(1 + x^2)$ is an example of a class of curves each of which is called a **witch of Agnesi**. Sketch the curve and find the tangent line to the curve at $x = 5$. (The word *witch* here is a mistranslation of the original Italian, as described at

http://mathworld.wolfram.com/WitchofAgnesi.html

and

http://instructional1.calstatela.edu/sgray/Agnesi/WitchHistory/Historynamewitch.html.)

⇒

9. If $f'(4) = 5$, $g'(4) = 12$, $(fg)(4) = f(4)g(4) = 2$, and $g(4) = 6$, compute $f(4)$ and $\dfrac{d}{dx}\dfrac{f}{g}$ at 4. ⇒

3.5 The Chain Rule

So far we have seen how to compute the derivative of a function built up from other functions by addition, subtraction, multiplication and division. There is another very important way that we combine simple functions to make more complicated functions: function composition, as discussed in section 2.3. For example, consider $\sqrt{625 - x^2}$. This function has many simpler components, like 625 and x^2, and then there is that square root symbol, so the square root function $\sqrt{x} = x^{1/2}$ is involved. The obvious question is: can we compute the derivative using the derivatives of the constituents $625 - x^2$ and $\sqrt{x}$? We can indeed. In general, if $f(x)$ and $g(x)$ are functions, we can compute the derivatives of $f(g(x))$ and $g(f(x))$ in terms of $f'(x)$ and $g'(x)$.

EXAMPLE 3.5.1 Form the two possible compositions of $f(x) = \sqrt{x}$ and $g(x) = 625 - x^2$ and compute the derivatives. First, $f(g(x)) = \sqrt{625 - x^2}$, and the derivative is $-x/\sqrt{625 - x^2}$ as we have seen. Second, $g(f(x)) = 625 - (\sqrt{x})^2 = 625 - x$ with derivative -1. Of course, these calculations do not use anything new, and in particular the derivative of $f(g(x))$ was somewhat tedious to compute from the definition. □

Suppose we want the derivative of $f(g(x))$. Again, let's set up the derivative and play some algebraic tricks:

$$\begin{aligned}\frac{d}{dx} f(g(x)) &= \lim_{\Delta x \to 0} \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(g(x + \Delta x)) - f(g(x))}{g(x + \Delta x)) - g(x)} \frac{g(x + \Delta x)) - g(x)}{\Delta x}\end{aligned}$$

Now we see immediately that the second fraction turns into $g'(x)$ when we take the limit. The first fraction is more complicated, but it too looks something like a derivative. The denominator, $g(x + \Delta x)) - g(x)$, is a change in the value of g, so let's abbreviate it as

$\Delta g = g(x + \Delta x)) - g(x)$, which also means $g(x + \Delta x) = g(x) + \Delta g$. This gives us

$$\lim_{\Delta x \to 0} \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g}.$$

As Δx goes to 0, it is also true that Δg goes to 0, because $g(x + \Delta x)$ goes to $g(x)$. So we can rewrite this limit as

$$\lim_{\Delta g \to 0} \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g}.$$

Now this looks exactly like a derivative, namely $f'(g(x))$, that is, the function $f'(x)$ with x replaced by $g(x)$. If this all withstands scrutiny, we then get

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

Unfortunately, there is a small flaw in the argument. Recall that what we mean by $\lim_{\Delta x \to 0}$ involves what happens when Δx is close to 0 *but not equal to 0.* The qualification is very important, since we must be able to divide by Δx. But when Δx is close to 0 but not equal to 0, $\Delta g = g(x + \Delta x)) - g(x)$ is close to 0 *and possibly equal to 0.* This means it doesn't really make sense to divide by Δg. Fortunately, it is possible to recast the argument to avoid this difficulty, but it is a bit tricky; we will not include the details, which can be found in many calculus books. Note that many functions g do have the property that $g(x + \Delta x) - g(x) \neq 0$ when Δx is small, and for these functions the argument above is fine.

The chain rule has a particularly simple expression if we use the Leibniz notation for the derivative. The quantity $f'(g(x))$ is the derivative of f with x replaced by g; this can be written df/dg. As usual, $g'(x) = dg/dx$. Then the chain rule becomes

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}.$$

This looks like trivial arithmetic, but it is not: dg/dx is not a fraction, that is, not literal division, but a single symbol that means $g'(x)$. Nevertheless, it turns out that what looks like trivial arithmetic, and is therefore easy to remember, is really true.

It will take a bit of practice to make the use of the chain rule come naturally—it is more complicated than the earlier differentiation rules we have seen.

EXAMPLE 3.5.2 Compute the derivative of $\sqrt{625 - x^2}$. We already know that the answer is $-x/\sqrt{625 - x^2}$, computed directly from the limit. In the context of the chain rule, we have $f(x) = \sqrt{x}$, $g(x) = 625 - x^2$. We know that $f'(x) = (1/2)x^{-1/2}$, so

$f'(g(x)) = (1/2)(625 - x^2)^{-1/2}$. Note that this is a two step computation: first compute $f'(x)$, then replace x by $g(x)$. Since $g'(x) = -2x$ we have

$$f'(g(x))g'(x) = \frac{1}{2\sqrt{625 - x^2}}(-2x) = \frac{-x}{\sqrt{625 - x^2}}.$$

□

EXAMPLE 3.5.3 Compute the derivative of $1/\sqrt{625 - x^2}$. This is a quotient with a constant numerator, so we could use the quotient rule, but it is simpler to use the chain rule. The function is $(625-x^2)^{-1/2}$, the composition of $f(x) = x^{-1/2}$ and $g(x) = 625-x^2$. We compute $f'(x) = (-1/2)x^{-3/2}$ using the power rule, and then

$$f'(g(x))g'(x) = \frac{-1}{2(625 - x^2)^{3/2}}(-2x) = \frac{x}{(625 - x^2)^{3/2}}.$$

□

In practice, of course, you will need to use more than one of the rules we have developed to compute the derivative of a complicated function.

EXAMPLE 3.5.4 Compute the derivative of

$$f(x) = \frac{x^2 - 1}{x\sqrt{x^2 + 1}}.$$

The "last" operation here is division, so to get started we need to use the quotient rule first. This gives

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1)'x\sqrt{x^2 + 1} - (x^2 - 1)(x\sqrt{x^2 + 1})'}{x^2(x^2 + 1)} \\ &= \frac{2x^2\sqrt{x^2 + 1} - (x^2 - 1)(x\sqrt{x^2 + 1})'}{x^2(x^2 + 1)}. \end{aligned}$$

Now we need to compute the derivative of $x\sqrt{x^2 + 1}$. This is a product, so we use the product rule:

$$\frac{d}{dx}x\sqrt{x^2 + 1} = x\frac{d}{dx}\sqrt{x^2 + 1} + \sqrt{x^2 + 1}.$$

Finally, we use the chain rule:

$$\frac{d}{dx}\sqrt{x^2 + 1} = \frac{d}{dx}(x^2 + 1)^{1/2} = \frac{1}{2}(x^2 + 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 1}}.$$

And putting it all together:

$$f'(x) = \frac{2x^2\sqrt{x^2+1} - (x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}.$$

$$= \frac{2x^2\sqrt{x^2+1} - (x^2-1)\left(x\frac{x}{\sqrt{x^2+1}} + \sqrt{x^2+1}\right)}{x^2(x^2+1)}.$$

This can be simplified of course, but we have done all the calculus, so that only algebra is left. □

EXAMPLE 3.5.5 Compute the derivative of $\sqrt{1+\sqrt{1+\sqrt{x}}}$. Here we have a more complicated chain of compositions, so we use the chain rule twice. At the outermost "layer" we have the function $g(x) = 1+\sqrt{1+\sqrt{x}}$ plugged into $f(x) = \sqrt{x}$, so applying the chain rule once gives

$$\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} = \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\frac{d}{dx}\left(1+\sqrt{1+\sqrt{x}}\right).$$

Now we need the derivative of $\sqrt{1+\sqrt{x}}$. Using the chain rule again:

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}.$$

So the original derivative is

$$\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} = \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}.$$

$$= \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}$$

□

Using the chain rule, the power rule, and the product rule, it is possible to avoid using the quotient rule entirely.

EXAMPLE 3.5.6 Compute the derivative of $f(x) = \dfrac{x^3}{x^2+1}$. Write $f(x) = x^3(x^2+1)^{-1}$, then

$$\begin{aligned} f'(x) &= x^3\frac{d}{dx}(x^2+1)^{-1} + 3x^2(x^2+1)^{-1} \\ &= x^3(-1)(x^2+1)^{-2}(2x) + 3x^2(x^2+1)^{-1} \\ &= -2x^4(x^2+1)^{-2} + 3x^2(x^2+1)^{-1} \\ &= \frac{-2x^4}{(x^2+1)^2} + \frac{3x^2}{x^2+1} \\ &= \frac{-2x^4}{(x^2+1)^2} + \frac{3x^2(x^2+1)}{(x^2+1)^2} \\ &= \frac{-2x^4+3x^4+3x^2}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2} \end{aligned}$$

Note that we already had the derivative on the second line; all the rest is simplification. It is easier to get to this answer by using the quotient rule, so there's a trade off: more work for fewer memorized formulas. □

Exercises 3.5.

Find the derivatives of the functions. For extra practice, and to check your answers, do some of these in more than one way if possible.

1. $x^4 - 3x^3 + (1/2)x^2 + 7x - \pi$ ⇒

2. $x^3 - 2x^2 + 4\sqrt{x}$ ⇒

3. $(x^2+1)^3$ ⇒

4. $x\sqrt{169-x^2}$ ⇒

5. $(x^2-4x+5)\sqrt{25-x^2}$ ⇒

6. $\sqrt{r^2-x^2}$, r is a constant ⇒

7. $\sqrt{1+x^4}$ ⇒

8. $\dfrac{1}{\sqrt{5-\sqrt{x}}}$. ⇒

9. $(1+3x)^2$ ⇒

10. $\dfrac{(x^2+x+1)}{(1-x)}$ ⇒

11. $\dfrac{\sqrt{25-x^2}}{x}$ ⇒

12. $\sqrt{\dfrac{169}{x} - x}$ ⇒

13. $\sqrt{x^3 - x^2 - (1/x)}$ ⇒

14. $100/(100-x^2)^{3/2}$ ⇒

15. $\sqrt[3]{x+x^3}$ ⇒

16. $\sqrt{(x^2+1)^2 + \sqrt{1+(x^2+1)^2}}$ ⇒

17. $(x+8)^5$ ⇒

18. $(4-x)^3$ ⇒

19. $(x^2+5)^3$ ⇒

20. $(6-2x^2)^3$ ⇒

21. $(1-4x^3)^{-2}$ ⇒

22. $5(x+1-1/x)$ ⇒

23. $4(2x^2-x+3)^{-2}$ ⇒

24. $\dfrac{1}{1+1/x}$ ⇒

25. $\dfrac{-3}{4x^2-2x+1}$ ⇒

26. $(x^2+1)(5-2x)/2$ ⇒

27. $(3x^2+1)(2x-4)^3$ ⇒

28. $\dfrac{x+1}{x-1}$ ⇒

29. $\dfrac{x^2-1}{x^2+1}$ ⇒

30. $\dfrac{(x-1)(x-2)}{x-3}$ ⇒

31. $\dfrac{2x^{-1}-x^{-2}}{3x^{-1}-4x^{-2}}$ ⇒

32. $3(x^2+1)(2x^2-1)(2x+3)$ ⇒

33. $\dfrac{1}{(2x+1)(x-3)}$ ⇒

34. $((2x+1)^{-1}+3)^{-1}$ ⇒

35. $(2x+1)^3(x^2+1)^2$ ⇒

36. Find an equation for the tangent line to $f(x) = (x-2)^{1/3}/(x^3+4x-1)^2$ at $x = 1$. ⇒

37. Find an equation for the tangent line to $y = 9x^{-2}$ at $(3, 1)$. ⇒

38. Find an equation for the tangent line to $(x^2-4x+5)\sqrt{25-x^2}$ at $(3, 8)$. ⇒

39. Find an equation for the tangent line to $\dfrac{(x^2+x+1)}{(1-x)}$ at $(2, -7)$. ⇒

40. Find an equation for the tangent line to $\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$ at $(1, \sqrt{4+\sqrt{5}})$. ⇒

4

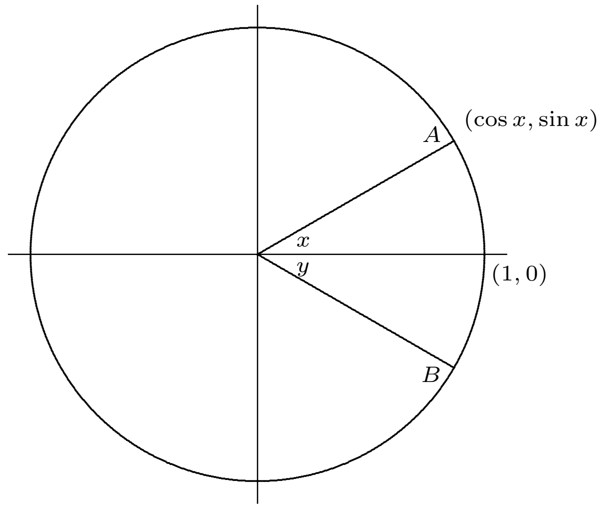
Transcendental Functions

So far we have used only algebraic functions as examples when finding derivatives, that is, functions that can be built up by the usual algebraic operations of addition, subtraction, multiplication, division, and raising to constant powers. Both in theory and practice there are other functions, called transcendental, that are very useful. Most important among these are the trigonometric functions, the inverse trigonometric functions, exponential functions, and logarithms.

4.1 Trigonometric Functions

When you first encountered the trigonometric functions it was probably in the context of "triangle trigonometry," defining, for example, the sine of an angle as the "side opposite over the hypotenuse." While this will still be useful in an informal way, we need to use a more expansive definition of the trigonometric functions. First an important note: while degree measure of angles is sometimes convenient because it is so familiar, it turns out to be ill-suited to mathematical calculation, so (almost) everything we do will be in terms of **radian measure** of angles.

To define the radian measurement system, we consider the unit circle in the xy-plane:

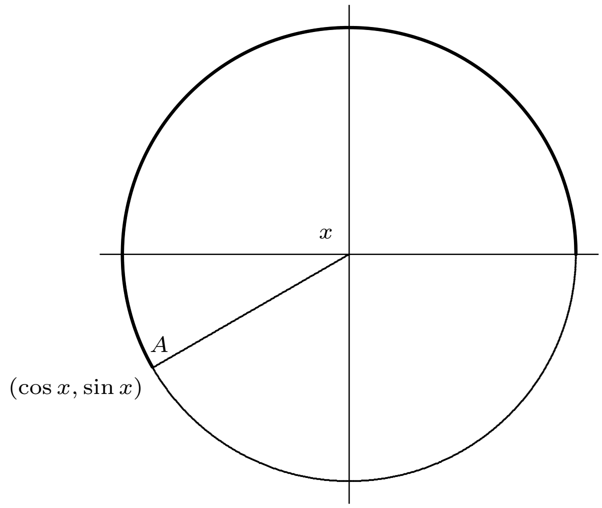


An angle, x, at the center of the circle is associated with an arc of the circle which is said to **subtend** the angle. In the figure, this arc is the portion of the circle from point $(1,0)$ to point A. The length of this arc is the radian measure of the angle x; the fact that the radian measure is an actual geometric length is largely responsible for the usefulness of radian measure. The circumference of the unit circle is $2\pi r = 2\pi(1) = 2\pi$, so the radian measure of the full circular angle (that is, of the 360 degree angle) is 2π.

While an angle with a particular measure can appear anywhere around the circle, we need a fixed, conventional location so that we can use the coordinate system to define properties of the angle. The standard convention is to place the starting radius for the angle on the positive x-axis, and to measure positive angles counterclockwise around the circle. In the figure, x is the standard location of the angle $\pi/6$, that is, the length of the arc from $(1,0)$ to A is $\pi/6$. The angle y in the picture is $-\pi/6$, because the distance from $(1,0)$ to B along the circle is also $\pi/6$, but in a clockwise direction.

Now the fundamental trigonometric definitions are: the cosine of x and the sine of x are the first and second coordinates of the point A, as indicated in the figure. The angle x shown can be viewed as an angle of a right triangle, meaning the usual triangle definitions of the sine and cosine also make sense. Since the hypotenuse of the triangle is 1, the "side opposite over hypotenuse" definition of the sine is the second coordinate of point A over 1, which is just the second coordinate; in other words, both methods give the same value for the sine.

The simple triangle definitions work only for angles that can "fit" in a right triangle, namely, angles between 0 and $\pi/2$. The coordinate definitions, on the other hand, apply to any angles, as indicated in this figure:



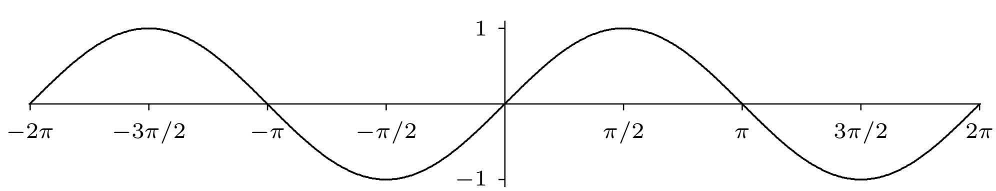
The angle x is subtended by the heavy arc in the figure, that is, $x = 7\pi/6$. Both coordinates of point A in this figure are negative, so the sine and cosine of $7\pi/6$ are both negative.

The remaining trigonometric functions can be most easily defined in terms of the sine and cosine, as usual:

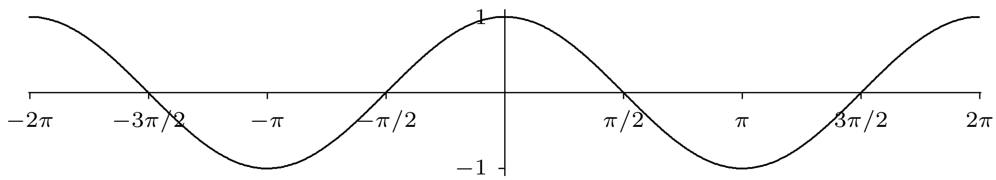
$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\csc x = \frac{1}{\sin x}$$

and they can also be defined as the corresponding ratios of coordinates.

Although the trigonometric functions are defined in terms of the unit circle, the unit circle diagram is not what we normally consider the graph of a trigonometric function. (The unit circle is the graph of, well, the circle.) We can easily get a qualitatively correct idea of the graphs of the trigonometric functions from the unit circle diagram. Consider the sine function, $y = \sin x$. As x increases from 0 in the unit circle diagram, the second coordinate of the point A goes from 0 to a maximum of 1, then back to 0, then to a minimum of -1, then back to 0, and then it obviously repeats itself. So the graph of $y = \sin x$ must look something like this:



Similarly, as angle x increases from 0 in the unit circle diagram, the first coordinate of the point A goes from 1 to 0 then to -1, back to 0 and back to 1, so the graph of $y = \cos x$ must look something like this:



Exercises 4.1.

Some useful trigonometric identities are in appendix B.

1. Find all values of θ such that $\sin(\theta) = -1$; give your answer in radians. ⇒

2. Find all values of θ such that $\cos(2\theta) = 1/2$; give your answer in radians. ⇒

3. Use an angle sum identity to compute $\cos(\pi/12)$. ⇒

4. Use an angle sum identity to compute $\tan(5\pi/12)$. ⇒

5. Verify the identity $\cos^2(t)/(1 - \sin(t)) = 1 + \sin(t)$.

6. Verify the identity $2\csc(2\theta) = \sec(\theta)\csc(\theta)$.

7. Verify the identity $\sin(3\theta) - \sin(\theta) = 2\cos(2\theta)\sin(\theta)$.

8. Sketch $y = 2\sin(x)$.

9. Sketch $y = \sin(3x)$.

10. Sketch $y = \sin(-x)$.

11. Find all of the solutions of $2\sin(t) - 1 - \sin^2(t) = 0$ in the interval $[0, 2\pi]$. ⇒

4.2 The Derivative of $\sin x$

What about the derivative of the sine function? The rules for derivatives that we have are no help, since $\sin x$ is not an algebraic function. We need to return to the definition of the derivative, set up a limit, and try to compute it. Here's the definition:

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}.$$

Using some trigonometric identities, we can make a little progress on the quotient:

$$\begin{aligned}\frac{\sin(x + \Delta x) - \sin x}{\Delta x} &= \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x} \\ &= \sin x \frac{\cos \Delta x - 1}{\Delta x} + \cos x \frac{\sin \Delta x}{\Delta x}.\end{aligned}$$

This isolates the difficult bits in the two limits

$$\lim_{\Delta x\to 0}\frac{\cos\Delta x-1}{\Delta x}\quad\text{and}\quad\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}.$$

Here we get a little lucky: it turns out that once we know the second limit the first is quite easy. The second is quite tricky, however. Indeed, it is the hardest limit we will actually compute, and we devote a section to it.

4.3 A hard limit

We want to compute this limit:

$$\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}.$$

Equivalently, to make the notation a bit simpler, we can compute

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

In the original context we need to keep x and Δx separate, but here it doesn't hurt to rename Δx to something more convenient.

To do this we need to be quite clever, and to employ some indirect reasoning. The indirect reasoning is embodied in a theorem, frequently called the **squeeze theorem**.

THEOREM 4.3.1 Squeeze Theorem Suppose that $g(x)\le f(x)\le h(x)$ for all x close to a but not equal to a. If $\lim_{x\to a} g(x)=L=\lim_{x\to a} h(x)$, then $\lim_{x\to a} f(x)=L$. ■

This theorem can be proved using the official definition of limit. We won't prove it here, but point out that it is easy to understand and believe graphically. The condition says that $f(x)$ is trapped between $g(x)$ below and $h(x)$ above, and that at $x=a$, both g and h approach the same value. This means the situation looks something like figure 4.3.1. The wiggly curve is $x^2\sin(\pi/x)$, the upper and lower curves are x^2 and $-x^2$. Since the sine function is always between -1 and 1, $-x^2\le x^2\sin(\pi/x)\le x^2$, and it is easy to see that $\lim_{x\to 0} -x^2=0=\lim_{x\to 0} x^2$. It is not so easy to see directly, that is algebraically, that $\lim_{x\to 0} x^2\sin(\pi/x)=0$, because the π/x prevents us from simply plugging in $x=0$. The squeeze theorem makes this "hard limit" as easy as the trivial limits involving x^2.

To do the hard limit that we want, $\lim_{x\to 0}(\sin x)/x$, we will find two simpler functions g and h so that $g(x)\le(\sin x)/x\le h(x)$, and so that $\lim_{x\to 0} g(x)=\lim_{x\to 0} h(x)$. Not too surprisingly, this will require some trigonometry and geometry. Referring to figure 4.3.2, x is the measure of the angle in radians. Since the circle has radius 1, the coordinates of

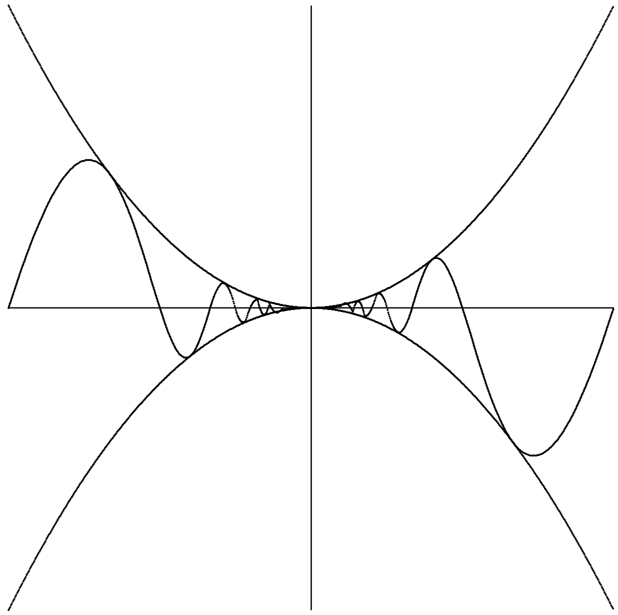


Figure 4.3.1 The squeeze theorem.

point A are $(\cos x, \sin x)$, and the area of the small triangle is $(\cos x \sin x)/2$. This triangle is completely contained within the circular wedge-shaped region bordered by two lines and the circle from $(1,0)$ to point A. Comparing the areas of the triangle and the wedge we see $(\cos x \sin x)/2 \le x/2$, since the area of a circular region with angle θ and radius r is $\theta r^2/2$. With a little algebra this turns into $(\sin x)/x \le 1/\cos x$, giving us the h we seek.

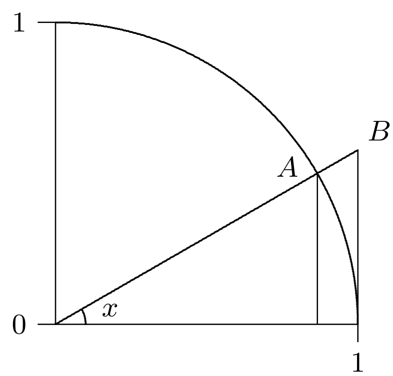


Figure 4.3.2 Visualizing $\sin x/x$.

To find g, we note that the circular wedge is completely contained inside the larger triangle. The height of the triangle, from $(1,0)$ to point B, is $\tan x$, so comparing areas we get $x/2 \le (\tan x)/2 = \sin x/(2\cos x)$. With a little algebra this becomes $\cos x \le (\sin x)/x$. So now we have

$$\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}.$$

Finally, the two limits $\lim_{x\to 0}\cos x$ and $\lim_{x\to 0} 1/\cos x$ are easy, because $\cos(0) = 1$. By the squeeze theorem, $\lim_{x\to 0}(\sin x)/x = 1$ as well.

Before we can complete the calculation of the derivative of the sine, we need one other limit:

$$\lim_{x\to 0}\frac{\cos x - 1}{x}.$$

This limit is just as hard as $\sin x/x$, but closely related to it, so that we don't have to do a similar calculation; instead we can do a bit of tricky algebra.

$$\frac{\cos x - 1}{x} = \frac{\cos x - 1}{x}\frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{x(\cos x + 1)} = \frac{-\sin^2 x}{x(\cos x + 1)} = -\frac{\sin x}{x}\frac{\sin x}{\cos x + 1}.$$

To compute the desired limit it is sufficient to compute the limits of the two final fractions, as x goes to 0. The first of these is the hard limit we've just done, namely 1. The second turns out to be simple, because the denominator presents no problem:

$$\lim_{x\to 0}\frac{\sin x}{\cos x + 1} = \frac{\sin 0}{\cos 0 + 1} = \frac{0}{2} = 0.$$

Thus,

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Exercises 4.3.

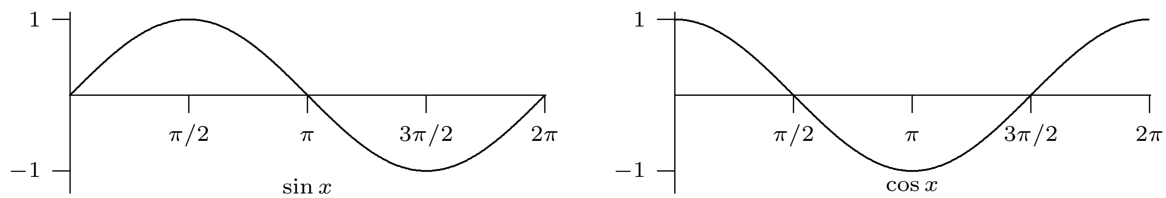
1. Compute $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{x}$ ⇒
2. Compute $\displaystyle\lim_{x\to 0}\frac{\sin(7x)}{\sin(2x)}$ ⇒
3. Compute $\displaystyle\lim_{x\to 0}\frac{\cot(4x)}{\csc(3x)}$ ⇒
4. Compute $\displaystyle\lim_{x\to 0}\frac{\tan x}{x}$ ⇒
5. Compute $\displaystyle\lim_{x\to \pi/4}\frac{\sin x - \cos x}{\cos(2x)}$ ⇒
6. For all $x \ge 0$, $4x - 9 \le f(x) \le x^2 - 4x + 7$. Find $\displaystyle\lim_{x\to 4} f(x)$. ⇒
7. For all x, $2x \le g(x) \le x^4 - x^2 + 2$. Find $\displaystyle\lim_{x\to 1} g(x)$. ⇒
8. Use the Squeeze Theorem to show that $\displaystyle\lim_{x\to 0} x^4\cos(2/x) = 0$.

4.4 The Derivative of $\sin x$, continued

Now we can complete the calculation of the derivative of the sine:

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\sin x\frac{\cos\Delta x-1}{\Delta x}+\cos x\frac{\sin\Delta x}{\Delta x}\\ &= \sin x\cdot 0+\cos x\cdot 1=\cos x.\end{aligned}$$

The derivative of a function measures the slope or steepness of the function; if we examine the graphs of the sine and cosine side by side, it should be that the latter appears to accurately describe the slope of the former, and indeed this is true:



Notice that where the cosine is zero the sine does appear to have a horizontal tangent line, and that the sine appears to be steepest where the cosine takes on its extreme values of 1 and −1.

Of course, now that we know the derivative of the sine, we can compute derivatives of more complicated functions involving the sine.

EXAMPLE 4.4.1 Compute the derivative of $\sin(x^2)$.

$$\frac{d}{dx}\sin(x^2)=\cos(x^2)\cdot 2x=2x\cos(x^2).$$

□

EXAMPLE 4.4.2 Compute the derivative of $\sin^2(x^3-5x)$.

$$\begin{aligned}\frac{d}{dx}\sin^2(x^3-5x) &= \frac{d}{dx}(\sin(x^3-5x))^2\\ &= 2(\sin(x^3-5x))^1\cos(x^3-5x)(3x^2-5)\\ &= 2(3x^2-5)\cos(x^3-5x)\sin(x^3-5x).\end{aligned}$$

□

Exercises 4.4.

Find the derivatives of the following functions.

1. $\sin^2(\sqrt{x})$ ⇒

2. $\sqrt{x}\sin x$ ⇒

3. $\dfrac{1}{\sin x}$ ⇒

4. $\dfrac{x^2+x}{\sin x}$ ⇒

5. $\sqrt{1-\sin^2 x}$ ⇒

4.5 Derivatives of the Trigonometric Functions

All of the other trigonometric functions can be expressed in terms of the sine, and so their derivatives can easily be calculated using the rules we already have. For the cosine we need to use two identities,

$$\cos x = \sin(x + \frac{\pi}{2}),$$
$$\sin x = -\cos(x + \frac{\pi}{2}).$$

Now:

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin(x+\frac{\pi}{2}) = \cos(x+\frac{\pi}{2})\cdot 1 = -\sin x$$
$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$
$$\frac{d}{dx}\sec x = \frac{d}{dx}(\cos x)^{-1} = -1(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

The derivatives of the cotangent and cosecant are similar and left as exercises.

Exercises 4.5.

Find the derivatives of the following functions.

1. $\sin x \cos x$ ⇒

2. $\sin(\cos x)$ ⇒

3. $\sqrt{x \tan x}$ ⇒

4. $\tan x/(1+\sin x)$ ⇒

5. $\cot x$ ⇒

6. $\csc x$ ⇒

7. $x^3\sin(23x^2)$ ⇒

8. $\sin^2 x + \cos^2 x$ ⇒

9. $\sin(\cos(6x))$ ⇒

10. Compute $\dfrac{d}{d\theta}\dfrac{\sec\theta}{1+\sec\theta}$. ⇒

11. Compute $\dfrac{d}{dt}t^5\cos(6t)$. ⇒

12. Compute $\dfrac{d}{dt}\dfrac{t^3\sin(3t)}{\cos(2t)}$. ⇒

13. Find all points on the graph of $f(x) = \sin^2(x)$ at which the tangent line is horizontal. ⇒

14. Find all points on the graph of $f(x) = 2\sin(x) - \sin^2(x)$ at which the tangent line is horizontal. ⇒

15. Find an equation for the tangent line to $\sin^2(x)$ at $x = \pi/3$. ⇒

16. Find an equation for the tangent line to $\sec^2 x$ at $x = \pi/3$. ⇒

17. Find an equation for the tangent line to $\cos^2 x - \sin^2(4x)$ at $x = \pi/6$. ⇒

18. Find the points on the curve $y = x + 2\cos x$ that have a horizontal tangent line. ⇒

19. Let C be a circle of radius r. Let A be an arc on C subtending a central angle θ. Let B be the chord of C whose endpoints are the endpoints of A. (Hence, B also subtends θ.) Let s be the length of A and let d be the length of B. Sketch a diagram of the situation and compute $\lim_{\theta \to 0^+} s/d$.

4.6 Exponential and Logarithmic functions

An exponential function has the form a^x, where a is a constant; examples are 2^x, 10^x, e^x. The logarithmic functions are the **inverses** of the exponential functions, that is, functions that "undo" the exponential functions, just as, for example, the cube root function "undoes" the cube function: $\sqrt[3]{2^3} = 2$. Note that the original function also undoes the inverse function: $(\sqrt[3]{8})^3 = 8$.

Let $f(x) = 2^x$. The inverse of this function is called the logarithm base 2, denoted $\log_2(x)$ or (especially in computer science circles) $\lg(x)$. What does this really mean? The logarithm must undo the action of the exponential function, so for example it must be that $\lg(2^3) = 3$—starting with 3, the exponential function produces $2^3 = 8$, and the logarithm of 8 must get us back to 3. A little thought shows that it is not a coincidence that $\lg(2^3)$ simply gives the exponent—the exponent *is* the original value that we must get back to. In other words, *the logarithm is the exponent.* Remember this catchphrase, and what it means, and you won't go wrong. (You *do* have to remember what it means. Like any good mnemonic, "the logarithm is the exponent" leaves out a lot of detail, like "Which exponent?" and "Exponent of what?")

EXAMPLE 4.6.1 What is the value of $\log_{10}(1000)$? The "10" tells us the appropriate number to use for the base of the exponential function. The logarithm is the exponent, so the question is, what exponent E makes $10^E = 1000$? If we can find such an E, then $\log_{10}(1000) = \log_{10}(10^E) = E$; finding the appropriate exponent is the same as finding the logarithm. In this case, of course, it is easy: $E = 3$ so $\log_{10}(1000) = 3$. □

Let's review some laws of exponents and logarithms; let a be a positive number. Since $a^5 = a\cdot a\cdot a\cdot a\cdot a$ and $a^3 = a\cdot a\cdot a$, it's clear that $a^5\cdot a^3 = a\cdot a\cdot a\cdot a\cdot a\cdot a\cdot a\cdot a = a^8 = a^{5+3}$, and in general that $a^m a^n = a^{m+n}$. Since "the logarithm is the exponent," it's no surprise that this translates directly into a fact about the logarithm function. Here are three facts from the example: $\log_a(a^5) = 5$, $\log_a(a^3) = 3$, $\log_a(a^8) = 8$. So $\log_a(a^5a^3) = \log_a(a^8) = 8 = 5 + 3 = \log_a(a^5) + \log_a(a^3)$. Now let's make this a bit more general. Suppose A and B are two numbers, $A = a^x$, and $B = a^y$. Then $\log_a(AB) = \log_a(a^xa^y) = \log_a(a^{x+y}) = x + y = \log_a(A) + \log_a(B)$.

Now consider $(a^5)^3 = a^5 \cdot a^5 \cdot a^5 = a^{5+5+5} = a^{5\cdot 3} = a^{15}$. Again it's clear that more generally $(a^m)^n = a^{mn}$, and again this gives us a fact about logarithms. If $A = a^x$ then $A^y = (a^x)^y = a^{xy}$, so $\log_a(A^y) = xy = y\log_a(A)$—the exponent can be "pulled out in front."

We have cheated a bit in the previous two paragraphs. It is obvious that $a^5 = a\cdot a\cdot a\cdot a\cdot a$ and $a^3 = a \cdot a \cdot a$ and that the rest of the example follows; likewise for the second example. But when we consider an exponential function a^x we can't be limited to substituting integers for x. What does $a^{2.5}$ or $a^{-1.3}$ or a^π mean? And is it really true that $a^{2.5}a^{-1.3} = a^{2.5-1.3}$? The answer to the first question is actually quite difficult, so we will evade it; the answer to the second question is "yes."

We'll evade the full answer to the hard question, but we have to know something about exponential functions. You need first to understand that since it's not "obvious" what 2^x should mean, we are really free to make it mean whatever we want, so long as we keep the behavior that *is* obvious, namely, when x is a positive integer. What else do we want to be true about 2^x? We want the properties of the previous two paragraphs to be true for all exponents: $2^x2^y = 2^{x+y}$ and $(2^x)^y = 2^{xy}$.

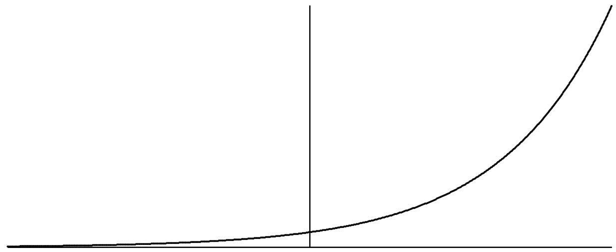
After the positive integers, the next easiest number to understand is 0: $2^0 = 1$. You have presumably learned this fact in the past; why is it true? It is true precisely because we want $2^a2^b = 2^{a+b}$ to be true about the function 2^x. We need it to be true that $2^02^x = 2^{0+x} = 2^x$, and this only works if $2^0 = 1$. The same argument implies that $a^0 = 1$ for any a.

The next easiest set of numbers to understand is the negative integers: for example, $2^{-3} = 1/2^3$. We know that whatever 2^{-3} means it must be that $2^{-3}2^3 = 2^{-3+3} = 2^0 = 1$, which means that 2^{-3} must be $1/2^3$. In fact, by the same argument, once we know what 2^x means for some value of x, 2^{-x} must be $1/2^x$ and more generally $a^{-x} = 1/a^x$.

Next, consider an exponent $1/q$, where q is a positive integer. We want it to be true that $(2^x)^y = 2^{xy}$, so $(2^{1/q})^q = 2$. This means that $2^{1/q}$ is a q-th root of 2, $2^{1/q} = \sqrt[q]{2}$. This is all we need to understand that $2^{p/q} = (2^{1/q})^p = (\sqrt[q]{2})^p$ and $a^{p/q} = (a^{1/q})^p = (\sqrt[q]{a})^p$.

What's left is the hard part: what does 2^x mean when x cannot be written as a fraction, like $x = \sqrt{2}$ or $x = \pi$? What we know so far is how to assign meaning to 2^x

whenever $x = p/q$; if we were to graph this we'd see something like this:



But this is a poor picture, because you can't see that the "curve" is really a whole lot of individual points, above the rational numbers on the x-axis. There are really a lot of "holes" in the curve, above $x = \pi$, for example. But (this is the hard part) it is possible to prove that the holes can be "filled in", and that the resulting function, called 2^x, really does have the properties we want, namely that $2^x 2^y = 2^{x+y}$ and $(2^x)^y = 2^{xy}$.

Exercises 4.6.

1. Expand $\log_{10}((x+45)^7(x-2))$.

2. Expand $\log_2 \dfrac{x^3}{3x-5+(7/x)}$.

3. Write $\log_2 3x + 17\log_2(x-2) - 2\log_2(x^2+4x+1)$ as a single logarithm.

4. Solve $\log_2(1+\sqrt{x}) = 6$ for x.

5. Solve $2^{x^2} = 8$ for x.

6. Solve $\log_2(\log_3(x)) = 1$ for x.

4.7 Derivatives of the exponential and logarithmic functions

As with the sine, we don't know anything about derivatives that allows us to compute the derivatives of the exponential and logarithmic functions without going back to basics. Let's do a little work with the definition again:

$$\begin{aligned}
\frac{d}{dx}a^x &= \lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{a^x a^{\Delta x}-a^x}{\Delta x}\\
&= \lim_{\Delta x\to 0} a^x\frac{a^{\Delta x}-1}{\Delta x}\\
&= a^x\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}
\end{aligned}$$

There are two interesting things to note here: As in the case of the sine function we are left with a limit that involves Δx but not x, which means that whatever $\lim_{\Delta x \to 0} (a^{\Delta x} - 1)/\Delta x$ is, we know that it is a number, that is, a constant. This means that a^x has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is $\lim_{x \to 0} \sin x/x = 1$; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that $\lim_{\Delta x \to 0} (a^{\Delta x} - 1)/\Delta x$ even exists—does this fraction really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider $(2^x - 1)/x$ for some small values of x: 1, 0.828427124, 0.756828460, 0.724061864, 0.70838051, 0.70070877 when x is 1, 1/2, 1/4, 1/8, 1/16, 1/32, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider next $(3^x - 1)/x$: 2, 1.464101616, 1.264296052, 1.177621520, 1.13720773, 1.11768854, at the same values of x. It turns out to be true that in the limit this is about 1.1. Two examples don't establish a pattern, but if you do more examples you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a = 2$ and $a = 3$ the limit will be exactly 1; the value at which this happens is called e, so that

$$\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1.$$

As you might guess from our two examples, e is closer to 3 than to 2, and in fact $e \approx 2.718$.

Now we see that the function e^x has a truly remarkable property:

$$\begin{aligned} \frac{d}{dx} e^x &= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{e^x e^{\Delta x} - e^x}{\Delta x} \\ &= \lim_{\Delta x \to 0} e^x \frac{e^{\Delta x} - 1}{\Delta x} \\ &= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} \\ &= e^x \end{aligned}$$

That is, e^x is its own derivative, or in other words the slope of e^x is the same as its height, or the same as its second coordinate: The function $f(x) = e^x$ goes through the point (z, e^z) and has slope e^z there, no matter what z is. It is sometimes convenient to express the function e^x without an exponent, since complicated exponents can be hard to read. In such cases we use $\exp(x)$, e.g., $\exp(1 + x^2)$ instead of e^{1+x^2}.

What about the logarithm function? This too is hard, but as the cosine function was easier to do once the sine was done, so the logarithm is easier to do now that we know the derivative of the exponential function. Let's start with $\log_e x$, which as you probably know is often abbreviated $\ln x$ and called the "natural logarithm" function.

Consider the relationship between the two functions, namely, that they are inverses, that one "undoes" the other. Graphically this means that they have the same graph except that one is "flipped" or "reflected" through the line $y = x$, as shown in figure 4.7.1.

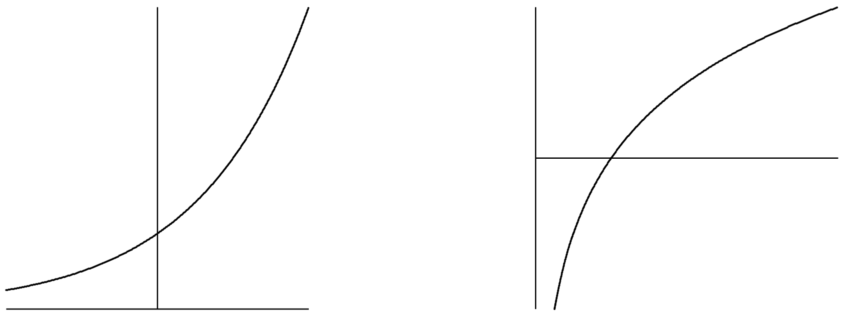


Figure 4.7.1 The exponential and logarithm functions.

This means that the slopes of these two functions are closely related as well: For example, the slope of e^x is e at $x = 1$; at the corresponding point on the $\ln(x)$ curve, the slope must be $1/e$, because the "rise" and the "run" have been interchanged. Since the slope of e^x is e at the point $(1, e)$, the slope of $\ln(x)$ is $1/e$ at the point $(e, 1)$.

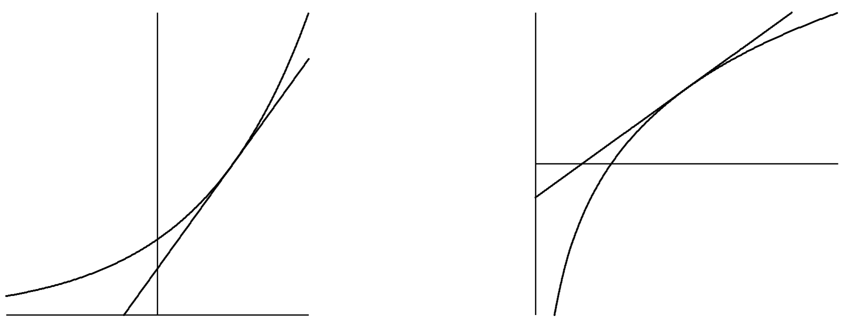


Figure 4.7.2 Slope of the exponential and logarithm functions.

More generally, we know that the slope of e^x is e^z at the point (z, e^z), so the slope of $\ln(x)$ is $1/e^z$ at (e^z, z), as indicated in figure 4.7.2. In other words, the slope of $\ln x$ is the reciprocal of the first coordinate at any point; this means that the slope of $\ln x$ at $(x, \ln x)$ is $1/x$. The upshot is:

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof—it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon.

Note that $\ln x$ is defined only for $x > 0$. It is sometimes useful to consider the function $\ln|x|$, a function defined for $x \neq 0$. When $x < 0$, $\ln|x| = \ln(-x)$ and

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Thus whether x is positive or negative, the derivative is the same.

What about the functions a^x and $\log_a x$? We know that the derivative of a^x is some constant times a^x itself, but what constant? Remember that "the logarithm is the exponent" and you will see that $a = e^{\ln a}$. Then

$$a^x = (e^{\ln a})^x = e^{x\ln a},$$

and we can compute the derivative using the chain rule:

$$\frac{d}{dx}a^x = \frac{d}{dx}(e^{\ln a})^x = \frac{d}{dx}e^{x\ln a} = (\ln a)e^{x\ln a} = (\ln a)a^x.$$

The constant is simply $\ln a$. Likewise we can compute the derivative of the logarithm function $\log_a x$. Since

$$x = e^{\ln x}$$

we can take the logarithm base a of both sides to get

$$\log_a(x) = \log_a(e^{\ln x}) = \ln x \log_a e.$$

Then

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e.$$

This is a perfectly good answer, but we can improve it slightly. Since

$$\begin{aligned} a &= e^{\ln a} \\ \log_a(a) &= \log_a(e^{\ln a}) = \ln a \log_a e \\ 1 &= \ln a \log_a e \\ \frac{1}{\ln a} &= \log_a e, \end{aligned}$$

we can replace $\log_a e$ to get

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

You may if you wish memorize the formulas

$$\frac{d}{dx}a^x = (\ln a)a^x \quad \text{and} \quad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

Because the "trick" $a = e^{\ln a}$ is often useful, and sometimes essential, it may be better to remember the trick, not the formula.

EXAMPLE 4.7.1 Compute the derivative of $f(x) = 2^x$.

$$\begin{aligned}
\frac{d}{dx}2^x &= \frac{d}{dx}(e^{\ln 2})^x \\
&= \frac{d}{dx}e^{x\ln 2} \\
&= \left(\frac{d}{dx}x\ln 2\right)e^{x\ln 2} \\
&= (\ln 2)e^{x\ln 2} = 2^x\ln 2
\end{aligned}$$

□

EXAMPLE 4.7.2 Compute the derivative of $f(x) = 2^{x^2} = 2^{(x^2)}$.

$$\begin{aligned}
\frac{d}{dx}2^{x^2} &= \frac{d}{dx}e^{x^2\ln 2} \\
&= \left(\frac{d}{dx}x^2\ln 2\right)e^{x^2\ln 2} \\
&= (2\ln 2)xe^{x^2\ln 2} \\
&= (2\ln 2)x2^{x^2}
\end{aligned}$$

□

EXAMPLE 4.7.3 Compute the derivative of $f(x) = x^x$. At first this appears to be a new kind of function: it is not a constant power of x, and it does not seem to be an exponential function, since the base is not constant. But in fact it is no harder than the previous example.

$$\begin{aligned}
\frac{d}{dx}x^x &= \frac{d}{dx}e^{x\ln x} \\
&= \left(\frac{d}{dx}x\ln x\right)e^{x\ln x} \\
&= (x\frac{1}{x} + \ln x)x^x \\
&= (1 + \ln x)x^x
\end{aligned}$$

□

EXAMPLE 4.7.4 Recall that we have not justified the power rule except when the exponent is a positive or negative integer. We can use the exponential function to take care of other exponents.

$$\begin{aligned}\frac{d}{dx}x^r &= \frac{d}{dx}e^{r\ln x}\\ &= \left(\frac{d}{dx}r\ln x\right)e^{r\ln x}\\ &= (r\frac{1}{x})x^r\\ &= rx^{r-1}\end{aligned}$$

□

Exercises 4.7.

In 1–19, find the derivatives of the functions.

1. 3^{x^2} ⇒

2. $\dfrac{\sin x}{e^x}$ ⇒

3. $(e^x)^2$ ⇒

4. $\sin(e^x)$ ⇒

5. $e^{\sin x}$ ⇒

6. $x^{\sin x}$ ⇒

7. x^3e^x ⇒

8. $x+2^x$ ⇒

9. $(1/3)^{x^2}$ ⇒

10. e^{4x}/x ⇒

11. $\ln(x^3+3x)$ ⇒

12. $\ln(\cos(x))$ ⇒

13. $\sqrt{\ln(x^2)}/x$ ⇒

14. $\ln(\sec(x)+\tan(x))$ ⇒

15. $x^{\cos(x)}$ ⇒

16. $x\ln x$

17. $\ln(\ln(3x))$

18. $\dfrac{1+\ln(3x^2)}{1+\ln(4x)}$

19. $\dfrac{x^8(x-23)^{1/2}}{27x^6(4x-6)^8}$

20. Find the value of a so that the tangent line to $y=\ln(x)$ at $x=a$ is a line through the origin. Sketch the resulting situation. ⇒

21. If $f(x)=\ln(x^3+2)$ compute $f'(e^{1/3})$.

4.8 Implicit Differentiation

As we have seen, there is a close relationship between the derivatives of e^x and $\ln x$ because these functions are inverses. Rather than relying on pictures for our understanding, we would like to be able to exploit this relationship computationally. In fact this technique can help us find derivatives in many situations, not just when we seek the derivative of an inverse function.

We will begin by illustrating the technique to find what we already know, the derivative of $\ln x$. Let's write $y = \ln x$ and then $x = e^{\ln x} = e^y$, that is, $x = e^y$. We say that this equation defines the function $y = \ln x$ implicitly because while it is not an explicit expression $y = \ldots$, it is true that if $x = e^y$ then y is in fact the natural logarithm function. Now, for the time being, pretend that all we know of y is that $x = e^y$; what can we say about derivatives? We can take the derivative of both sides of the equation:

$$\frac{d}{dx}x = \frac{d}{dx}e^y.$$

Then using the chain rule on the right hand side:

$$1 = \left(\frac{d}{dx}y\right)e^y = y'e^y.$$

Then we can solve for y':

$$y' = \frac{1}{e^y} = \frac{1}{x}.$$

There is one little difficulty here. To use the chain rule to compute $d/dx(e^y) = y'e^y$ we need to know that the function y *has* a derivative. All we have shown is that *if* it has a derivative then that derivative must be $1/x$. When using this method we will always have to assume that the desired derivative exists, but fortunately this is a safe assumption for most such problems.

The example $y = \ln x$ involved an inverse function defined implicitly, but other functions can be defined implicitly, and sometimes a single equation can be used to implicitly define more than one function. Here's a familiar example. The equation $r^2 = x^2 + y^2$ describes a circle of radius r. The circle is not a function $y = f(x)$ because for some values of x there are two corresponding values of y. If we want to work with a function, we can break the circle into two pieces, the upper and lower semicircles, each of which is a function. Let's call these $y = U(x)$ and $y = L(x)$; in fact this is a fairly simple example, and it's possible to give explicit expressions for these: $U(x) = \sqrt{r^2 - x^2}$ and $L(x) = -\sqrt{r^2 - x^2}$. But it's somewhat easier, and quite useful, to view both functions as given implicitly by $r^2 = x^2 + y^2$: both $r^2 = x^2 + U(x)^2$ and $r^2 = x^2 + L(x)^2$ are true, and we can think of $r^2 = x^2 + y^2$ as defining both $U(x)$ and $L(x)$.

Now we can take the derivative of both sides as before, remembering that y is not simply a variable but a function—in this case, y is either $U(x)$ or $L(x)$ but we're not yet specifying which one. When we take the derivative we just have to remember to apply the chain rule where y appears.

$$\begin{aligned}\frac{d}{dx}r^2 &= \frac{d}{dx}(x^2+y^2)\\ 0 &= 2x+2yy'\\ y' &= \frac{-2x}{2y} = -\frac{x}{y}\end{aligned}$$

Now we have an expression for y', but it contains y as well as x. This means that if we want to compute y' for some particular value of x we'll have to know or compute y at that value of x as well. It is at this point that we will need to know whether y is $U(x)$ or $L(x)$. Occasionally it will turn out that we can avoid explicit use of $U(x)$ or $L(x)$ by the nature of the problem

EXAMPLE 4.8.1 Find the slope of the circle $4 = x^2 + y^2$ at the point $(1, -\sqrt{3})$. Since we know both the x and y coordinates of the point of interest, we do not need to explicitly recognize that this point is on $L(x)$, and we do not need to use $L(x)$ to compute y—but we could. Using the calculation of y' from above,

$$y' = -\frac{x}{y} = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

It is instructive to compare this approach to others.

We might have recognized at the start that $(1, -\sqrt{3})$ is on the function $y = L(x) = -\sqrt{4-x^2}$. We could then take the derivative of $L(x)$, using the power rule and the chain rule, to get

$$L'(x) = -\frac{1}{2}(4-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{4-x^2}}.$$

Then we could compute $L'(1) = 1/\sqrt{3}$ by substituting $x = 1$.

Alternately, we could realize that the point is on $L(x)$, but use the fact that $y' = -x/y$. Since the point is on $L(x)$ we can replace y by $L(x)$ to get

$$y' = -\frac{x}{L(x)} = \frac{x}{\sqrt{4-x^2}},$$

without computing the derivative of $L(x)$ explicitly. Then we substitute $x = 1$ and get the same answer as before. □

In the case of the circle it is possible to find the functions $U(x)$ and $L(x)$ explicitly, but there are potential advantages to using implicit differentiation anyway. In some cases it is more difficult or impossible to find an explicit formula for y and implicit differentiation is the only way to find the derivative.

EXAMPLE 4.8.2 Find the derivative of any function defined implicitly by $yx^2+e^y=x$. We treat y as an unspecified function and use the chain rule:

$$\begin{aligned}
\frac{d}{dx}(yx^2+e^y) &= \frac{d}{dx}x \\
(y\cdot 2x + y'\cdot x^2) + y'e^y &= 1 \\
y'x^2 + y'e^y &= 1-2xy \\
y'(x^2+e^y) &= 1-2xy \\
y' &= \frac{1-2xy}{x^2+e^y}
\end{aligned}$$

□

You might think that the step in which we solve for y' could sometimes be difficult—after all, we're using implicit differentiation here because we can't solve the equation $yx^2+e^y=x$ for y, so maybe after taking the derivative we get something that is hard to solve for y'. In fact, *this never happens.* All occurrences y' come from applying the chain rule, and whenever the chain rule is used it deposits a single y' multiplied by some other expression. So it will always be possible to group the terms containing y' together and factor out the y', just as in the previous example. If you ever get anything more difficult you have made a mistake and should fix it before trying to continue.

It is sometimes the case that a situation leads naturally to an equation that defines a function implicitly.

EXAMPLE 4.8.3 Consider all the points (x,y) that have the property that the distance from (x,y) to (x_1,y_1) plus the distance from (x,y) to (x_2,y_2) is $2a$ (a is some constant). These points form an ellipse, which like a circle is not a function but can viewed as two functions pasted together. Because we know how to write down the distance between two points, we can write down an implicit equation for the ellipse:

$$\sqrt{(x-x_1)^2+(y-y_1)^2}+\sqrt{(x-x_2)^2+(y-y_2)^2}=2a.$$

Then we can use implicit differentiation to find the slope of the ellipse at any point, though the computation is rather messy. □

EXAMPLE 4.8.4 We have already justified the power rule by using the exponential function, but we could also do it for rational exponents by using implicit differentiation. Suppose that $y=x^{m/n}$, where m and n are positive integers. We can write this implicitly as $y^n=x^m$, then because we justified the power rule for integers, we can take the derivative of each side:

$$
\begin{aligned}
ny^{n-1}y' &= mx^{m-1} \\
y' &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} \\
y' &= \frac{m}{n}\frac{x^{m-1}}{(x^{m/n})^{n-1}} \\
y' &= \frac{m}{n}x^{m-1-(m/n)(n-1)} \\
y' &= \frac{m}{n}x^{m-1-m+(m/n)} \\
y' &= \frac{m}{n}x^{(m/n)-1}
\end{aligned}
$$

□

Exercises 4.8.

In exercises 1–8, find a formula for the derivative y' at the point (x, y):

1. $y^2 = 1 + x^2$ ⇒
2. $x^2 + xy + y^2 = 7$ ⇒
3. $x^3 + xy^2 = y^3 + yx^2$ ⇒
4. $4 \cos x \sin y = 1$ ⇒
5. $\sqrt{x} + \sqrt{y} = 9$ ⇒
6. $\tan(x/y) = x + y$ ⇒
7. $\sin(x + y) = xy$ ⇒
8. $\frac{1}{x} + \frac{1}{y} = 7$ ⇒
9. A hyperbola passing through $(8, 6)$ consists of all points whose distance from the origin is a constant more than its distance from the point (5,2). Find the slope of the tangent line to the hyperbola at $(8, 6)$. ⇒
10. Compute y' for the ellipse of example 4.8.3.
11. If $y = \log_a x$ then $a^y = x$. Use implicit differentiation to find y'.
12. The graph of the equation $x^2 - xy + y^2 = 9$ is an ellipse. Find the lines tangent to this curve at the two points where it intersects the x-axis. Show that these lines are parallel. ⇒
13. Repeat the previous problem for the points at which the ellipse intersects the y-axis. ⇒
14. Find the points on the ellipse from the previous two problems where the slope is horizontal and where it is vertical. ⇒
15. Find an equation for the tangent line to $x^4 = y^2 + x^2$ at $(2, \sqrt{12})$. (This curve is the **kampyle of Eudoxus**.) ⇒
16. Find an equation for the tangent line to $x^{2/3} + y^{2/3} = a^{2/3}$ at a point (x_1, y_1) on the curve, with $x_1 \neq 0$ and $y_1 \neq 0$. (This curve is an **astroid**.) ⇒
17. Find an equation for the tangent line to $(x^2 + y^2)^2 = x^2 - y^2$ at a point (x_1, y_1) on the curve, with $x_1 \neq 0, -1, 1$. (This curve is a **lemniscate**.) ⇒

Definition. Two curves are **orthogonal** if at each point of intersection, the angle between their tangent lines is $\pi/2$. Two families of curves, $\mathcal{A}$ and $\mathcal{B}$, are **orthogonal trajectories** of each other if given any curve C in $\mathcal{A}$ and any curve D in $\mathcal{B}$ the curves C and D are orthogonal. For example, the family of horizontal lines in the plane is orthogonal to the family of vertical lines in the plane.

18. Show that $x^2 - y^2 = 5$ is orthogonal to $4x^2 + 9y^2 = 72$. (Hint: You need to find the intersection points of the two curves and then show that the product of the derivatives at each intersection point is -1.)

19. Show that $x^2 + y^2 = r^2$ is orthogonal to $y = mx$. Conclude that the family of circles centered at the origin is an orthogonal trajectory of the family of lines that pass through the origin.

Note that there is a technical issue when $m = 0$. The circles fail to be differentiable when they cross the x-axis. However, the circles are orthogonal to the x-axis. Explain why. Likewise, the vertical line through the origin requires a separate argument.

20. For $k \neq 0$ and $c \neq 0$ show that $y^2 - x^2 = k$ is orthogonal to $yx = c$. In the case where k and c are both zero, the curves intersect at the origin. Are the curves $y^2 - x^2 = 0$ and $yx = 0$ orthogonal to each other?

21. Suppose that $m \neq 0$. Show that the family of curves $\{y = mx + b \mid b \in \mathbb{R}\}$ is orthogonal to the family of curves $\{y = -(x/m) + c \mid c \in \mathbb{R}\}$.

4.9 Inverse Trigonometric Functions

The trigonometric functions frequently arise in problems, and often it is necessary to invert the functions, for example, to find an angle with a specified sine. Of course, there are many angles with the same sine, so the sine function doesn't actually have an inverse that reliably "undoes" the sine function. If you know that $\sin x = 0.5$, you can't reverse this to discover x, that is, you can't solve for x, as there are infinitely many angles with sine 0.5. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values of the sine exactly once. If we "discard" all other angles, the resulting function does have a proper inverse.

The sine takes on all values between -1 and 1 exactly once on the interval $[-\pi/2, \pi/2]$. If we truncate the sine, keeping only the interval $[-\pi/2, \pi/2]$, as shown in figure 4.9.1, then this truncated sine has an inverse function. We call this the inverse sine or the arcsine, and write $y = \arcsin(x)$.

Recall that a function and its inverse undo each other in either order, for example, $(\sqrt[3]{x})^3 = x$ and $\sqrt[3]{x^3} = x$. This does not work with the sine and the "inverse sine" because the inverse sine is the inverse of the truncated sine function, not the real sine function. It is true that $\sin(\arcsin(x)) = x$, that is, the sine undoes the arcsine. It is not true that the arcsine undoes the sine, for example, $\sin(5\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$, so doing first the sine then the arcsine does not get us back where we started. This is because $5\pi/6$

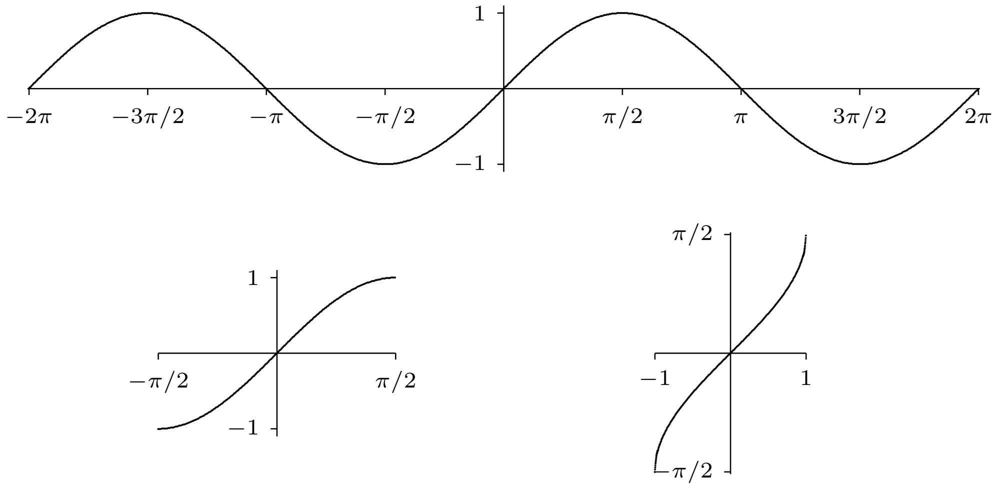


Figure 4.9.1 The sine, the truncated sine, the inverse sine.

is not in the domain of the truncated sine. If we start with an angle between $-\pi/2$ and $\pi/2$ then the arcsine does reverse the sine: $\sin(\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$.

What is the derivative of the arcsine? Since this is an inverse function, we can discover the derivative by using implicit differentiation. Suppose $y = \arcsin(x)$. Then

$$\sin(y) = \sin(\arcsin(x)) = x.$$

Now taking the derivative of both sides, we get

$$\begin{aligned} y' \cos y &= 1 \\ y' &= \frac{1}{\cos y} \end{aligned}$$

As we expect when using implicit differentiation, y appears on the right hand side here. We would certainly prefer to have y' written in terms of x, and as in the case of $\ln x$ we can actually do that here. Since $\sin^2 y + \cos^2 y = 1$, $\cos^2 y = 1 - \sin^2 y = 1 - x^2$. So $\cos y = \pm\sqrt{1 - x^2}$, but which is it—plus or minus? It could in general be either, but this isn't "in general": since $y = \arcsin(x)$ we know that $-\pi/2 \le y \le \pi/2$, and the cosine of an angle in this interval is always positive. Thus $\cos y = \sqrt{1 - x^2}$ and

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1 - x^2}}.$$

Note that this agrees with figure 4.9.1: the graph of the arcsine has positive slope everywhere.

We can do something similar for the cosine. As with the sine, we must first truncate the cosine so that it can be inverted, as shown in figure 4.9.2. Then we use implicit

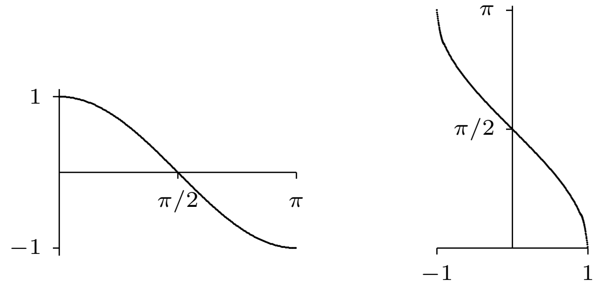


Figure 4.9.2 The truncated cosine, the inverse cosine.

differentiation to find that

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}.$$

Note that the truncated cosine uses a different interval than the truncated sine, so that if $y = \arccos(x)$ we know that $0 \le y \le \pi$. The computation of the derivative of the arccosine is left as an exercise.

Finally we look at the tangent; the other trigonometric functions also have "partial inverses" but the sine, cosine and tangent are enough for most purposes. The tangent, truncated tangent and inverse tangent are shown in figure 4.9.3; the derivative of the arctangent is left as an exercise.

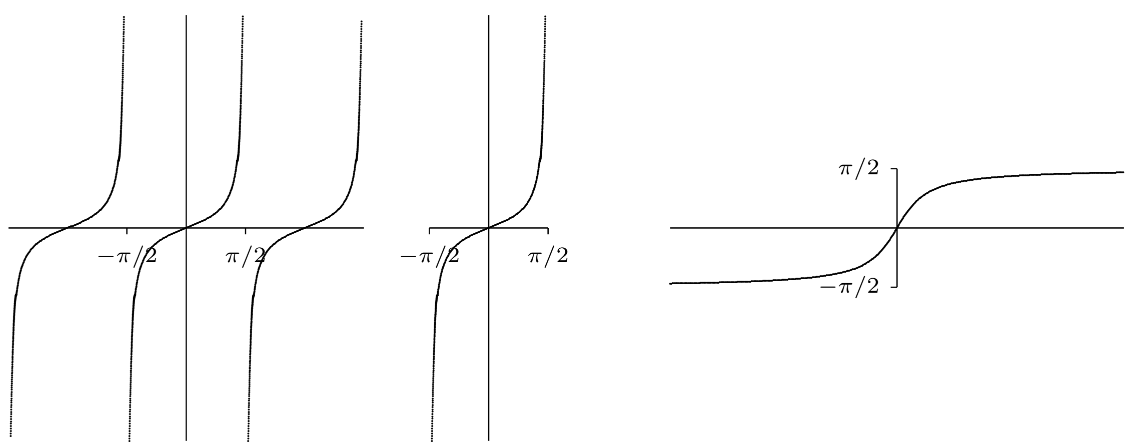


Figure 4.9.3 The tangent, the truncated tangent, the inverse tangent.

Exercises 4.9.

1. Show that the derivative of $\arccos x$ is $-\dfrac{1}{\sqrt{1-x^2}}$.

2. Show that the derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$.

3. The inverse of cot is usually defined so that the range of arccot is $(0, \pi)$. Sketch the graph of $y = \operatorname{arccot} x$. In the process you will make it clear what the domain of arccot is. Find the derivative of the arccotangent. ⇒

4. Show that $\operatorname{arccot} x + \arctan x = \pi/2$.

5. Find the derivative of $\arcsin(x^2)$. ⇒

6. Find the derivative of $\arctan(e^x)$. ⇒

7. Find the derivative of $\arccos(\sin x^3)$ ⇒

8. Find the derivative of $\ln((\arcsin x)^2)$ ⇒

9. Find the derivative of $\arccos e^x$ ⇒

10. Find the derivative of $\arcsin x + \arccos x$ ⇒

11. Find the derivative of $\log_5(\arctan(x^x))$ ⇒

5

Curve Sketching

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function and certain limits.

5.1 Maxima and Minima

A **local maximum point** on a function is a point (x, y) on the graph of the function whose y coordinate is larger than all other y coordinates on the graph at points "close to" (x, y). More precisely, $(x, f(x))$ is a local maximum if there is an interval (a, b) with $a < x < b$ and $f(x) \geq f(z)$ for every z in (a, b). Similarly, (x, y) is a **local minimum point** if it has locally the smallest y coordinate. Again being more precise: $(x, f(x))$ is a local minimum if there is an interval (a, b) with $a < x < b$ and $f(x) \leq f(z)$ for every z in (a, b). A **local extremum** is either a local minimum or a local maximum.

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well. Some examples of local maximum and minimum points are shown in figure 5.1.1.

If $(x, f(x))$ is a point where $f(x)$ reaches a local maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line must be horizontal. This is important enough to state as a theorem, though we will not prove it.

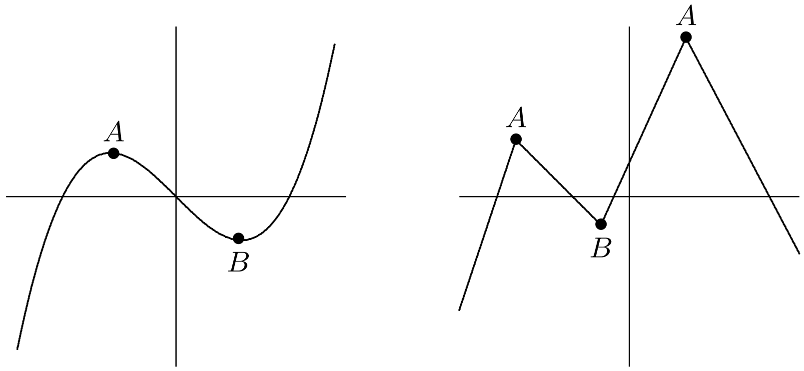


Figure 5.1.1 Some local maximum points (A) and minimum points (B).

THEOREM 5.1.1 Fermat's Theorem If $f(x)$ has a local extremum at $x = a$ and f is differentiable at a, then $f'(a) = 0$. ■

Thus, the only points at which a function can have a local maximum or minimum are points at which the derivative is zero, as in the left hand graph in figure 5.1.1, or the derivative is undefined, as in the right hand graph. Any value of x for which $f'(x)$ is zero or undefined is called a **critical value** for f. When looking for local maximum and minimum points, you are likely to make two sorts of mistakes: You may forget that a maximum or minimum can occur where the derivative does not exist, and so forget to check whether the derivative exists everywhere. You might also assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true. A portion of the graph of $f(x) = x^3$ is shown in figure 5.1.2. The derivative of f is $f'(x) = 3x^2$, and $f'(0) = 0$, but there is neither a maximum nor minimum at $(0, 0)$.

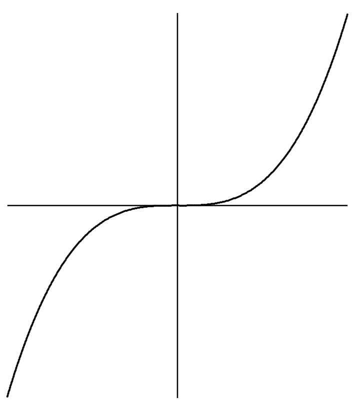


Figure 5.1.2 No maximum or minimum even though the derivative is zero.

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach, but one that is often tedious or difficult, is to test directly whether the y coordinates "near" the potential maximum or minimum are above or below the y coordinate at the point of interest. Of course, there are too many points "near" the point to test, but a little thought shows we need only test two provided we know that f is continuous (recall that this means that the graph of f has no jumps or gaps).

Suppose, for example, that we have identified three points at which f' is zero or nonexistent: (x_1, y_1), (x_2, y_2), (x_3, y_3), and $x_1 < x_2 < x_3$ (see figure 5.1.3). Suppose that we compute the value of $f(a)$ for $x_1 < a < x_2$, and that $f(a) < f(x_2)$. What can we say about the graph between a and x_2? Could there be a point $(b, f(b))$, $a < b < x_2$ with $f(b) > f(x_2)$? No: if there were, the graph would go up from $(a, f(a))$ to $(b, f(b))$ then down to $(x_2, f(x_2))$ and somewhere in between would have a local maximum point. (This is not obvious; it is a result of the Extreme Value Theorem, theorem 6.1.2.) But at that local maximum point the derivative of f would be zero or nonexistent, yet we already know that the derivative is zero or nonexistent only at x_1, x_2, and x_3. The upshot is that one computation tells us that $(x_2, f(x_2))$ has the largest y coordinate of any point on the graph near x_2 and to the left of x_2. We can perform the same test on the right. If we find that on both sides of x_2 the values are smaller, then there must be a local maximum at $(x_2, f(x_2))$; if we find that on both sides of x_2 the values are larger, then there must be a local minimum at $(x_2, f(x_2))$; if we find one of each, then there is neither a local maximum or minimum at x_2.

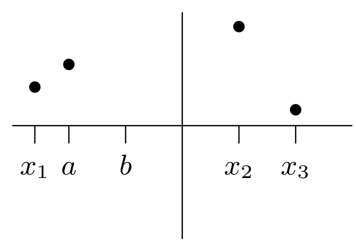


Figure 5.1.3 Testing for a maximum or minimum.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks—they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.

EXAMPLE 5.1.2 Find all local maximum and minimum points for the function $f(x) = x^3 - x$. The derivative is $f'(x) = 3x^2 - 1$. This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. Looking first at $x = \sqrt{3}/3$, we see that $f(\sqrt{3}/3) = -2\sqrt{3}/9$. Now we test two points on either side of $x = \sqrt{3}/3$, making sure that neither is farther away than the nearest critical value; since $\sqrt{3} < 3$, $\sqrt{3}/3 < 1$ and we can use $x = 0$ and $x = 1$. Since $f(0) = 0 > -2\sqrt{3}/9$ and $f(1) = 0 > -2\sqrt{3}/9$, there must be a local minimum at

$x = \sqrt{3}/3$. For $x = -\sqrt{3}/3$, we see that $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. This time we can use $x = 0$ and $x = -1$, and we find that $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, so there must be a local maximum at $x = -\sqrt{3}/3$. □

Of course this example is made very simple by our choice of points to test, namely $x = -1, 0, 1$. We could have used other values, say $-5/4$, $1/3$, and $3/4$, but this would have made the calculations considerably more tedious.

EXAMPLE 5.1.3 Find all local maximum and minimum points for $f(x) = \sin x + \cos x$. The derivative is $f'(x) = \cos x - \sin x$. This is always defined and is zero whenever $\cos x = \sin x$. Recalling that the $\cos x$ and $\sin x$ are the x and y coordinates of points on a unit circle, we see that $\cos x = \sin x$ when x is $\pi/4$, $\pi/4 \pm \pi$, $\pi/4 \pm 2\pi$, $\pi/4 \pm 3\pi$, etc. Since both sine and cosine have a period of 2π, we need only determine the status of $x = \pi/4$ and $x = 5\pi/4$. We can use 0 and $\pi/2$ to test the critical value $x = \pi/4$. We find that $f(\pi/4) = \sqrt{2}$, $f(0) = 1 < \sqrt{2}$ and $f(\pi/2) = 1$, so there is a local maximum when $x = \pi/4$ and also when $x = \pi/4 \pm 2\pi$, $\pi/4 \pm 4\pi$, etc. We can summarize this more neatly by saying that there are local maxima at $\pi/4 \pm 2k\pi$ for every integer k.

We use π and 2π to test the critical value $x = 5\pi/4$. The relevant values are $f(5\pi/4) = -\sqrt{2}$, $f(\pi) = -1 > -\sqrt{2}$, $f(2\pi) = 1 > -\sqrt{2}$, so there is a local minimum at $x = 5\pi/4$, $5\pi/4 \pm 2\pi$, $5\pi/4 \pm 4\pi$, etc. More succinctly, there are local minima at $5\pi/4 \pm 2k\pi$ for every integer k. □

Exercises 5.1.

In problems 1–12, find all local maximum and minimum points (x, y) by the method of this section.

1. $y = x^2 - x$ ⇒

2. $y = 2 + 3x - x^3$ ⇒

3. $y = x^3 - 9x^2 + 24x$ ⇒

4. $y = x^4 - 2x^2 + 3$ ⇒

5. $y = 3x^4 - 4x^3$ ⇒

6. $y = (x^2 - 1)/x$ ⇒

7. $y = 3x^2 - (1/x^2)$ ⇒

8. $y = \cos(2x) - x$ ⇒

9. $f(x) = \begin{cases} x - 1 & x < 2 \\ x^2 & x \geq 2 \end{cases}$ ⇒

10. $f(x) = \begin{cases} x - 3 & x < 3 \\ x^3 & 3 \leq x \leq 5 \\ 1/x & x > 5 \end{cases}$ ⇒

11. $f(x) = x^2 - 98x + 4$ ⇒

12. $f(x) = \begin{cases} -2 & x = 0 \\ 1/x^2 & x \neq 0 \end{cases}$ ⇒

13. For any real number x there is a unique integer n such that $n \leq x < n + 1$, and the greatest integer function is defined as $\lfloor x \rfloor = n$. Where are the critical values of the greatest integer function? Which are local maxima and which are local minima?

14. Explain why the function $f(x) = 1/x$ has no local maxima or minima.

15. How many critical points can a quadratic polynomial function have? ⇒

16. Show that a cubic polynomial can have at most two critical points. Give examples to show that a cubic polynomial can have zero, one, or two critical points.

17. Explore the family of functions $f(x) = x^3 + cx + 1$ where c is a constant. How many and what types of local extremes are there? Your answer should depend on the value of c, that is, different values of c will give different answers.

18. We generalize the preceding two questions. Let n be a positive integer and let f be a polynomial of degree n. How many critical points can f have? (Hint: Recall the **Fundamental Theorem of Algebra**, which says that a polynomial of degree n has at most n roots.)

5.2 The first derivative test

The method of the previous section for deciding whether there is a local maximum or minimum at a critical value is not always convenient. We can instead use information about the derivative $f'(x)$ to decide; since we have already had to compute the derivative to find the critical values, there is often relatively little extra work involved in this method.

How can the derivative tell us whether there is a maximum, minimum, or neither at a point? Suppose that $f'(a) = 0$. If there is a local maximum when $x = a$, the function must be lower near $x = a$ than it is right at $x = a$. If the derivative exists near $x = a$, this means $f'(x) > 0$ when x is near a and $x < a$, because the function must "slope up" just to the left of a. Similarly, $f'(x) < 0$ when x is near a and $x > a$, because f slopes down from the local maximum as we move to the right. Using the same reasoning, if there is a local minimum at $x = a$, the derivative of f must be negative just to the left of a and positive just to the right. If the derivative exists near a but does not change from positive to negative or negative to positive, that is, it is positive on both sides or negative on both sides, then there is neither a maximum nor minimum when $x = a$. See the first graph in figure 5.1.1 and the graph in figure 5.1.2 for examples.

EXAMPLE 5.2.1 Find all local maximum and minimum points for $f(x) = \sin x + \cos x$ using the first derivative test. The derivative is $f'(x) = \cos x - \sin x$ and from example 5.1.3 the critical values we need to consider are $\pi/4$ and $5\pi/4$.

The graphs of $\sin x$ and $\cos x$ are shown in figure 5.2.1. Just to the left of $\pi/4$ the cosine is larger than the sine, so $f'(x)$ is positive; just to the right the cosine is smaller than the sine, so $f'(x)$ is negative. This means there is a local maximum at $\pi/4$. Just to the left of $5\pi/4$ the cosine is smaller than the sine, and to the right the cosine is larger than the sine. This means that the derivative $f'(x)$ is negative to the left and positive to the right, so f has a local minimum at $5\pi/4$. □

Exercises 5.2.

In 1–13, find all critical points and identify them as local maximum points, local minimum points, or neither.

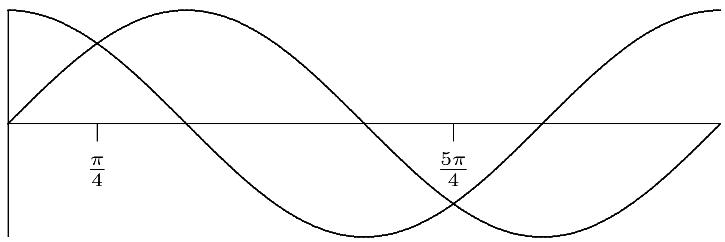


Figure 5.2.1 The sine and cosine.

1. $y = x^2 - x \Rightarrow$

2. $y = 2 + 3x - x^3 \Rightarrow$

3. $y = x^3 - 9x^2 + 24x \Rightarrow$

4. $y = x^4 - 2x^2 + 3 \Rightarrow$

5. $y = 3x^4 - 4x^3 \Rightarrow$

6. $y = (x^2 - 1)/x \Rightarrow$

7. $y = 3x^2 - (1/x^2) \Rightarrow$

8. $y = \cos(2x) - x \Rightarrow$

9. $f(x) = (5 - x)/(x + 2) \Rightarrow$

10. $f(x) = |x^2 - 121| \Rightarrow$

11. $f(x) = x^3/(x + 1) \Rightarrow$

12. $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

13. $f(x) = \sin^2 x \Rightarrow$

14. Find the maxima and minima of $f(x) = \sec x$. $\Rightarrow$

15. Let $f(\theta) = \cos^2(\theta) - 2\sin(\theta)$. Find the intervals where f is increasing and the intervals where f is decreasing in $[0, 2\pi]$. Use this information to classify the critical points of f as either local maximums, local minimums, or neither. $\Rightarrow$

16. Let $r > 0$. Find the local maxima and minima of the function $f(x) = \sqrt{r^2 - x^2}$ on its domain $[-r, r]$.

17. Let $f(x) = ax^2 + bx + c$ with $a \neq 0$. Show that f has exactly one critical point. Give conditions on a and b which guarantee that the critical point will be a maximum. It is possible to see this without using calculus at all; explain.

5.3 The second derivative test

derivative!secondThe basis of the first derivative test is that if the derivative changes from positive to negative at a point at which the derivative is zero then there is a local maximum at the point, and similarly for a local minimum. If f' changes from positive to negative it is decreasing; this means that the derivative of f', f'', might be negative, and if in fact f'' is negative then f' is definitely decreasing, so there is a local maximum at the point in question. Note well that f' might change from positive to negative while f'' is zero, in which case f'' gives us no information about the critical value. Similarly, if f' changes from negative to positive there is a local minimum at the point, and f' is increasing. If $f'' > 0$ at the point, this tells us that f' is increasing, and so there is a local minimum.

EXAMPLE 5.3.1 Consider again $f(x) = \sin x + \cos x$, with $f'(x) = \cos x - \sin x$ and $f''(x) = -\sin x - \cos x$. Since $f''(\pi/4) = -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2} < 0$, we know there is a local maximum at $\pi/4$. Since $f''(5\pi/4) = -- \sqrt{2}/2 - -\sqrt{2}/2 = \sqrt{2} > 0$, there is a local minimum at $5\pi/4$. □

When it works, the second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails, and sometimes the second derivative is quite difficult to evaluate; in such cases we must fall back on one of the previous tests.

EXAMPLE 5.3.2 Let $f(x) = x^4$. The derivatives are $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Zero is the only critical value, but $f''(0) = 0$, so the second derivative test tells us nothing. However, $f(x)$ is positive everywhere except at zero, so clearly $f(x)$ has a local minimum at zero. On the other hand, $f(x) = -x^4$ also has zero as its only critical value, and the second derivative is again zero, but $-x^4$ has a local maximum at zero. □

Exercises 5.3.

Find all local maximum and minimum points by the second derivative test.

1. $y = x^2 - x \Rightarrow$

2. $y = 2 + 3x - x^3 \Rightarrow$

3. $y = x^3 - 9x^2 + 24x \Rightarrow$

4. $y = x^4 - 2x^2 + 3 \Rightarrow$

5. $y = 3x^4 - 4x^3 \Rightarrow$

6. $y = (x^2 - 1)/x \Rightarrow$

7. $y = 3x^2 - (1/x^2) \Rightarrow$

8. $y = \cos(2x) - x \Rightarrow$

9. $y = 4x + \sqrt{1 - x} \Rightarrow$

10. $y = (x + 1)/\sqrt{5x^2 + 35} \Rightarrow$

11. $y = x^5 - x \Rightarrow$

12. $y = 6x + \sin 3x \Rightarrow$

13. $y = x + 1/x \Rightarrow$

14. $y = x^2 + 1/x \Rightarrow$

15. $y = (x + 5)^{1/4} \Rightarrow$

16. $y = \tan^2 x \Rightarrow$

17. $y = \cos^2 x - \sin^2 x \Rightarrow$

18. $y = \sin^3 x \Rightarrow$

5.4 Concavity and inflection points

We know that the sign of the derivative tells us whether a function is increasing or decreasing; for example, when $f'(x) > 0$, $f(x)$ is increasing. The sign of the second derivative $f''(x)$ tells us whether f' is increasing or decreasing; we have seen that if f' is zero and increasing at a point then there is a local minimum at the point, and if f' is zero and decreasing at a point then there is a local maximum at the point. Thus, we extracted information about f from information about f''.

We can get information from the sign of f'' even when f' is not zero. Suppose that $f''(a) > 0$. This means that near $x = a$, f' is increasing. If $f'(a) > 0$, this means that f slopes up and is getting steeper; if $f'(a) < 0$, this means that f slopes down and is getting

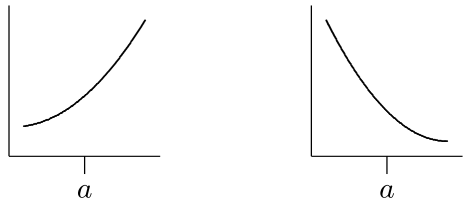


Figure 5.4.1 $f''(a) > 0$: $f'(a)$ positive and increasing, $f'(a)$ negative and increasing.

less steep. The two situations are shown in figure 5.4.1. A curve that is shaped like this is called **concave up**.

Now suppose that $f''(a) < 0$. This means that near $x = a$, f' is decreasing. If $f'(a) > 0$, this means that f slopes up and is getting less steep; if $f'(a) < 0$, this means that f slopes down and is getting steeper. The two situations are shown in figure 5.4.2. A curve that is shaped like this is called **concave down**.

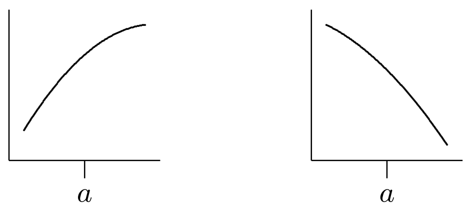


Figure 5.4.2 $f''(a) < 0$: $f'(a)$ positive and decreasing, $f'(a)$ negative and decreasing.

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. Of particular interest are points at which the concavity changes from up to down or down to up; such points are called **inflection points**. If the concavity changes from up to down at $x = a$, f'' changes from positive to the left of a to negative to the right of a, and usually $f''(a) = 0$. We can identify such points by first finding where $f''(x)$ is zero and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points. Note that it is possible that $f''(a) = 0$ but the concavity is the same on both sides; $f(x) = x^4$ at $x = 0$ is an example.

EXAMPLE 5.4.1 Describe the concavity of $f(x) = x^3 - x$. $f'(x) = 3x^2 - 1$, $f''(x) = 6x$. Since $f''(0) = 0$, there is potentially an inflection point at zero. Since $f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$ the concavity does change from down to up at zero, and the curve is concave down for all $x < 0$ and concave up for all $x > 0$. □

Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.

Exercises 5.4.

Describe the concavity of the functions in 1–18.

1. $y = x^2 - x$ ⇒

2. $y = 2 + 3x - x^3$ ⇒

3. $y = x^3 - 9x^2 + 24x$ ⇒

4. $y = x^4 - 2x^2 + 3$ ⇒

5. $y = 3x^4 - 4x^3$ ⇒

6. $y = (x^2 - 1)/x$ ⇒

7. $y = 3x^2 - (1/x^2)$ ⇒

8. $y = \sin x + \cos x$ ⇒

9. $y = 4x + \sqrt{1 - x}$ ⇒

10. $y = (x + 1)/\sqrt{5x^2 + 35}$ ⇒

11. $y = x^5 - x$ ⇒

12. $y = 6x + \sin 3x$ ⇒

13. $y = x + 1/x$ ⇒

14. $y = x^2 + 1/x$ ⇒

15. $y = (x + 5)^{1/4}$ ⇒

16. $y = \tan^2 x$ ⇒

17. $y = \cos^2 x - \sin^2 x$ ⇒

18. $y = \sin^3 x$ ⇒

19. Identify the intervals on which the graph of the function $f(x) = x^4 - 4x^3 + 10$ is of one of these four shapes: concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing. ⇒

20. Describe the concavity of $y = x^3 + bx^2 + cx + d$. You will need to consider different cases, depending on the values of the coefficients.

21. Let n be an integer greater than or equal to two, and suppose f is a polynomial of degree n. How many inflection points can f have? Hint: Use the second derivative test and the fundamental theorem of algebra.

5.5 Asymptotes and Other Things to Look For

A vertical asymptote is a place where the function becomes infinite, typically because the formula for the function has a denominator that becomes zero. For example, the reciprocal function $f(x) = 1/x$ has a vertical asymptote at $x = 0$, and the function $\tan x$ has a vertical asymptote at $x = \pi/2$ (and also at $x = -\pi/2$, $x = 3\pi/2$, etc.). Whenever the formula for a function contains a denominator it is worth looking for a vertical asymptote by checking to see if the denominator can ever be zero, and then checking the limit at such points. Note that there is not always a vertical asymptote where the denominator is zero: $f(x) = (\sin x)/x$ has a zero denominator at $x = 0$, but since $\lim_{x\to 0}(\sin x)/x = 1$ there is no asymptote there.

A horizontal asymptote is a horizontal line to which $f(x)$ gets closer and closer as x approaches ∞ (or as x approaches $-\infty$). For example, the reciprocal function has the x-axis for a horizontal asymptote. Horizontal asymptotes can be identified by computing the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Since $\lim_{x\to\infty} 1/x = \lim_{x\to-\infty} 1/x = 0$, the line $y = 0$ (that is, the x-axis) is a horizontal asymptote in both directions.

Some functions have asymptotes that are neither horizontal nor vertical, but some other line. Such asymptotes are somewhat more difficult to identify and we will ignore them.

If the domain of the function does not extend out to infinity, we should also ask what happens as x approaches the boundary of the domain. For example, the function $y = f(x) = 1/\sqrt{r^2 - x^2}$ has domain $-r < x < r$, and y becomes infinite as x approaches either r or $-r$. In this case we might also identify this behavior because when $x = \pm r$ the denominator of the function is zero.

If there are any points where the derivative fails to exist (a cusp or corner), then we should take special note of what the function does at such a point.

Finally, it is worthwhile to notice any symmetry. A function $f(x)$ that has the same value for $-x$ as for x, i.e., $f(-x) = f(x)$, is called an "even function." Its graph is symmetric with respect to the y-axis. Some examples of even functions are: x^n when n is an even number, $\cos x$, and $\sin^2 x$. On the other hand, a function that satisfies the property $f(-x) = -f(x)$ is called an "odd function." Its graph is symmetric with respect to the origin. Some examples of odd functions are: x^n when n is an odd number, $\sin x$, and $\tan x$. Of course, most functions are neither even nor odd, and do not have any particular symmetry.

Exercises 5.5.

Sketch the curves. Identify clearly any interesting features, including local maximum and minimum points, inflection points, asymptotes, and intercepts.

1. $y = x^5 - 5x^4 + 5x^3$

2. $y = x^3 - 3x^2 - 9x + 5$

3. $y = (x-1)^2(x+3)^{2/3}$

4. $x^2 + x^2y^2 = a^2y^2$, $a > 0$.

5. $y = xe^x$

6. $y = (e^x + e^{-x})/2$

7. $y = e^{-x}\cos x$

8. $y = e^x - \sin x$

9. $y = e^x/x$

10. $y = 4x + \sqrt{1-x}$

11. $y = (x+1)/\sqrt{5x^2+35}$

12. $y = x^5 - x$

13. $y = 6x + \sin 3x$

14. $y = x + 1/x$

15. $y = x^2 + 1/x$

16. $y = (x+5)^{1/4}$

17. $y = \tan^2 x$

18. $y = \cos^2 x - \sin^2 x$

19. $y = \sin^3 x$

20. $y = x(x^2+1)$

21. $y = x^3 + 6x^2 + 9x$

22. $y = x/(x^2-9)$

23. $y = x^2/(x^2+9)$

24. $y = 2\sqrt{x} - x$

25. $y = 3\sin(x) - \sin^3(x)$, for $x \in [0, 2\pi]$

26. $y = (x-1)/(x^2)$

For each of the following five functions, identify any vertical and horizontal asymptotes, and identify intervals on which the function is concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing.

27. $f(\theta) = \sec(\theta)$

28. $f(x) = 1/(1 + x^2)$

29. $f(x) = (x - 3)/(2x - 2)$

30. $f(x) = 1/(1 - x^2)$

31. $f(x) = 1 + 1/(x^2)$

32. Let $f(x) = 1/(x^2 - a^2)$, where $a \geq 0$. Find any vertical and horizontal asymptotes and the intervals upon which the given function is concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing. Discuss how the value of a affects these features.

6

Applications of the Derivative

6.1 Optimization

Many important applied problems involve finding the best way to accomplish some task. Often this involves finding the maximum or minimum value of some function: the minimum time to make a certain journey, the minimum cost for doing a task, the maximum power that can be generated by a device, and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

Generally such a problem will have the following mathematical form: Find the largest (or smallest) value of $f(x)$ when $a \le x \le b$. Sometimes a or b are infinite, but frequently the real world imposes some constraint on the values that x may have.

Such a problem differs in two ways from the local maximum and minimum problems we encountered when graphing functions: We are interested only in the function between a and b, and we want to know the largest or smallest value that $f(x)$ takes on, not merely values that are the largest or smallest in a small interval. That is, we seek not a local maximum or minimum but a **global** maximum or minimum, sometimes also called an **absolute** maximum or minimum.

Any global maximum or minimum must of course be a local maximum or minimum. If we find all possible local extrema, then the global maximum, *if it exists*, must be the largest of the local maxima and the global minimum, *if it exists*, must be the smallest of the local minima. We already know where local extrema can occur: only at those points at which $f'(x)$ is zero or undefined. Actually, there are two additional points at which a maximum or minimum can occur if the endpoints a and b are not infinite, namely, at a

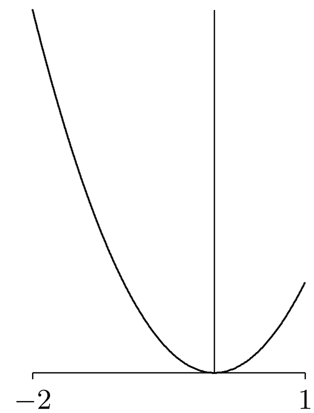


Figure 6.1.1 The function $f(x) = x^2$ restricted to $[-2, 1]$

and b. We have not previously considered such points because we have not been interested in limiting a function to a small interval. An example should make this clear.

EXAMPLE 6.1.1 Find the maximum and minimum values of $f(x) = x^2$ on the interval $[-2, 1]$, shown in figure 6.1.1. We compute $f'(x) = 2x$, which is zero at $x = 0$ and is always defined.

Since $f'(1) = 2$ we would not normally flag $x = 1$ as a point of interest, but it is clear from the graph that *when $f(x)$ is restricted to $[-2, 1]$ there is a local maximum at $x = 1$*. Likewise we would not normally pay attention to $x = -2$, but since we have truncated f at -2 we have introduced a new local maximum there as well. In a technical sense nothing new is going on here: When we truncate f we actually create a new function, let's call it g, that is defined only on the interval $[-2, 1]$. If we try to compute the derivative of this new function we actually find that it does not have a derivative at -2 or 1. Why? Because to compute the derivative at 1 we must compute the limit

$$\lim_{\Delta x \to 0} \frac{g(1 + \Delta x) - g(1)}{\Delta x}.$$

This limit does not exist because when $\Delta x > 0$, $g(1 + \Delta x)$ is not defined. It is simpler, however, simply to remember that we must always check the endpoints.

So the function g, that is, f restricted to $[-2, 1]$, has one critical value and two finite endpoints, any of which might be the global maximum or minimum. We could first determine which of these are local maximum or minimum points (or neither); then the largest local maximum must be the global maximum and the smallest local minimum must be the global minimum. It is usually easier, however, to compute the value of f at every point at which the global maximum or minimum might occur; the largest of these is the global maximum, the smallest is the global minimum.

So we compute $f(-2) = 4$, $f(0) = 0$, $f(1) = 1$. The global maximum is 4 at $x = -2$ and the global minimum is 0 at $x = 0$. □

It is possible that there is no global maximum or minimum. It is difficult, and not particularly useful, to express a complete procedure for determining whether this is the case. Generally, the best approach is to gain enough understanding of the shape of the graph to decide. Fortunately, only a rough idea of the shape is usually needed.

There are some particularly nice cases that are easy. A continuous function on a closed interval $[a,b]$ *always* has both a global maximum and a global minimum, so examining the critical values and the endpoints is enough:

THEOREM 6.1.2 Extreme value theorem If f is continuous on a closed interval $[a,b]$, then it has both a minimum and a maximum point. That is, there are real numbers c and d in $[a,b]$ so that for every x in $[a,b]$, $f(x) \le f(c)$ and $f(x) \ge f(d)$. ■

Another easy case: If a function is continuous and has a single critical value, then if there is a local maximum at the critical value it is a global maximum, and if it is a local minimum it is a global minimum. There may also be a global minimum in the first case, or a global maximum in the second case, but that will generally require more effort to determine.

EXAMPLE 6.1.3 Let $f(x) = -x^2 + 4x - 3$. Find the maximum value of $f(x)$ on the interval $[0,4]$. First note that $f'(x) = -2x+4 = 0$ when $x = 2$, and $f(2) = 1$. Next observe that $f'(x)$ is defined for all x, so there are no other critical values. Finally, $f(0) = -3$ and $f(4) = -3$. The largest value of $f(x)$ on the interval $[0,4]$ is $f(2) = 1$. □

EXAMPLE 6.1.4 Let $f(x) = -x^2 + 4x - 3$. Find the maximum value of $f(x)$ on the interval $[-1,1]$.

First note that $f'(x) = -2x + 4 = 0$ when $x = 2$. But $x = 2$ is not in the interval, so we don't use it. Thus the only two points to be checked are the endpoints; $f(-1) = -8$ and $f(1) = 0$. So the largest value of $f(x)$ on $[-1,1]$ is $f(1) = 0$. □

EXAMPLE 6.1.5 Find the maximum and minimum values of the function $f(x) = 7 + |x-2|$ for x between 1 and 4 inclusive. The derivative $f'(x)$ is never zero, but $f'(x)$ is undefined at $x = 2$, so we compute $f(2) = 7$. Checking the end points we get $f(1) = 8$ and $f(4) = 9$. The smallest of these numbers is $f(2) = 7$, which is, therefore, the minimum value of $f(x)$ on the interval $1 \le x \le 4$, and the maximum is $f(4) = 9$. □

EXAMPLE 6.1.6 Find all local maxima and minima for $f(x) = x^3 - x$, and determine whether there is a global maximum or minimum on the open interval $(-2,2)$. In example 5.1.2 we found a local maximum at $(-\sqrt{3}/3, 2\sqrt{3}/9)$ and a local minimum at $(\sqrt{3}/3, -2\sqrt{3}/9)$. Since the endpoints are not in the interval $(-2,2)$ they cannot be con-

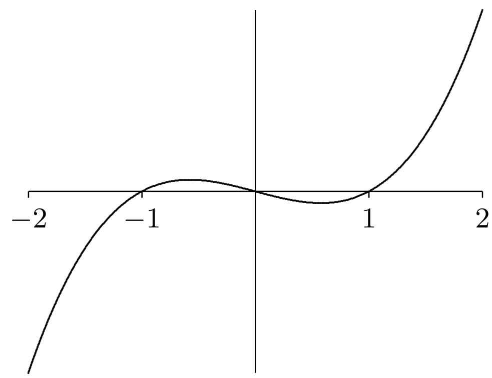


Figure 6.1.2 $f(x) = x^3 - x$

sidered. Is the lone local maximum a global maximum? Here we must look more closely at the graph. We know that on the closed interval $[-\sqrt{3}/3, \sqrt{3}/3]$ there is a global maximum at $x = -\sqrt{3}/3$ and a global minimum at $x = \sqrt{3}/3$. So the question becomes: what happens between -2 and $-\sqrt{3}/3$, and between $\sqrt{3}/3$ and 2? Since there is a local minimum at $x = \sqrt{3}/3$, the graph must continue up to the right, since there are no more critical values. This means no value of f will be less than $-2\sqrt{3}/9$ between $\sqrt{3}/3$ and 2, but it says nothing about whether we might find a value larger than the local maximum $2\sqrt{3}/9$. How can we tell? Since the function increases to the right of $\sqrt{3}/3$, we need to know what the function values do "close to" 2. Here the easiest test is to pick a number and do a computation to get some idea of what's going on. Since $f(1.9) = 4.959 > 2\sqrt{3}/9$, there is no global maximum at $-\sqrt{3}/3$, and hence no global maximum at all. (How can we tell that $4.959 > 2\sqrt{3}/9$? We can use a calculator to approximate the right hand side; if it is not even close to 4.959 we can take this as decisive. Since $2\sqrt{3}/9 \approx 0.3849$, there's really no question. Funny things can happen in the rounding done by computers and calculators, however, so we might be a little more careful, especially if the values come out quite close. In this case we can convert the relation $4.959 > 2\sqrt{3}/9$ into $(9/2)4.959 > \sqrt{3}$ and ask whether this is true. Since the left side is clearly larger than $4 \cdot 4$ which is clearly larger than $\sqrt{3}$, this settles the question.)

A similar analysis shows that there is also no global minimum. The graph of $f(x)$ on $(-2, 2)$ is shown in figure 6.1.2. □

EXAMPLE 6.1.7 Of all rectangles of area 100, which has the smallest perimeter?

First we must translate this into a purely mathematical problem in which we want to find the minimum value of a function. If x denotes one of the sides of the rectangle, then the adjacent side must be $100/x$ (in order that the area be 100). So the function we want to minimize is

$$f(x) = 2x + 2\frac{100}{x}$$

since the perimeter is twice the length plus twice the width of the rectangle. Not all values of x make sense in this problem: lengths of sides of rectangles must be positive, so $x > 0$. If $x > 0$ then so is $100/x$, so we need no second condition on x.

We next find $f'(x)$ and set it equal to zero: $0 = f'(x) = 2 - 200/x^2$. Solving $f'(x) = 0$ for x gives us $x = \pm 10$. We are interested only in $x > 0$, so only the value $x = 10$ is of interest. Since $f'(x)$ is defined everywhere on the interval $(0, \infty)$, there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at $x = 10$? The second derivative is $f''(x) = 400/x^3$, and $f''(10) > 0$, so there is a local minimum. Since there is only one critical value, this is also the global minimum, so the rectangle with smallest perimeter is the 10×10 square. □

EXAMPLE 6.1.8 You want to sell a certain number n of items in order to maximize your profit. Market research tells you that if you set the price at \$1.50, you will be able to sell 5000 items, and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total \$2000, and the per item cost of production ("marginal cost") is \$0.50. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

The first step is to convert the problem into a function maximization problem. Since we want to maximize profit by setting the price per item, we should look for a function $P(x)$ representing the profit when the price per item is x. Profit is revenue minus costs, and revenue is number of items sold times the price per item, so we get $P = nx - 2000 - 0.50n$. The number of items sold is itself a function of x, $n = 5000 + 1000(1.5 - x)/0.10$, because $(1.5 - x)/0.10$ is the number of multiples of 10 cents that the price is below \$1.50. Now we substitute for n in the profit function:

$$\begin{aligned} P(x) &= (5000 + 1000(1.5 - x)/0.10)x - 2000 - 0.5(5000 + 1000(1.5 - x)/0.10) \\ &= -10000x^2 + 25000x - 12000 \end{aligned}$$

We want to know the maximum value of this function when x is between 0 and 1.5. The derivative is $P'(x) = -20000x + 25000$, which is zero when $x = 1.25$. Since $P''(x) = -20000 < 0$, there must be a local maximum at $x = 1.25$, and since this is the only critical value it must be a global maximum as well. (Alternately, we could compute $P(0) = -12000$, $P(1.25) = 3625$, and $P(1.5) = 3000$ and note that $P(1.25)$ is the maximum of these.) Thus the maximum profit is \$3625, attained when we set the price at \$1.25 and sell 7500 items. □

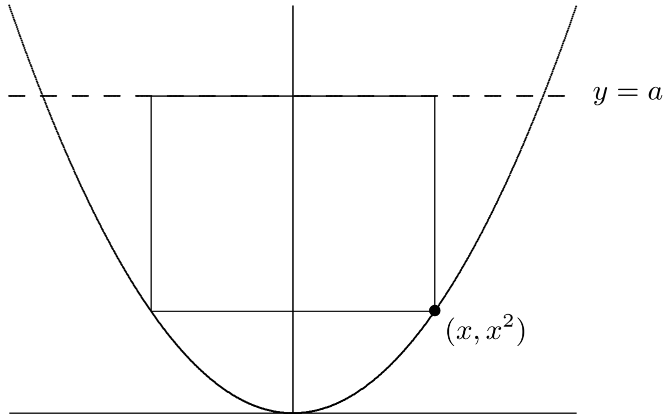


Figure 6.1.3 Rectangle in a parabola.

EXAMPLE 6.1.9 Find the largest rectangle (that is, the rectangle with largest area) that fits inside the graph of the parabola $y = x^2$ below the line $y = a$ (a is an unspecified constant value), with the top side of the rectangle on the horizontal line $y = a$; see figure 6.1.3.)

We want to find the maximum value of some function $A(x)$ representing area. Perhaps the hardest part of this problem is deciding what x should represent. The lower right corner of the rectangle is at (x, x^2), and once this is chosen the rectangle is completely determined. So we can let the x in $A(x)$ be the x of the parabola $f(x) = x^2$. Then the area is $A(x) = (2x)(a - x^2) = -2x^3 + 2ax$. We want the maximum value of $A(x)$ when x is in $[0, \sqrt{a}]$. (You might object to allowing $x = 0$ or $x = \sqrt{a}$, since then the "rectangle" has either no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area.)

Setting $0 = A'(x) = -6x^2 + 2a$ we get $x = \sqrt{a/3}$ as the only critical value. Testing this and the two endpoints, we have $A(0) = A(\sqrt{a}) = 0$ and $A(\sqrt{a/3}) = (4/9)\sqrt{3}a^{3/2}$. The maximum area thus occurs when the rectangle has dimensions $2\sqrt{a/3} \times (2/3)a$. □

EXAMPLE 6.1.10 If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)

Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: $\pi r^2 h/3$. Here R is a fixed value, but r and h can vary. Namely, we could choose r to be as large as possible—equal to R—by taking the height equal to R; or we could make the cone's height h larger at the expense of making r a little less than R. See the cross-section depicted in

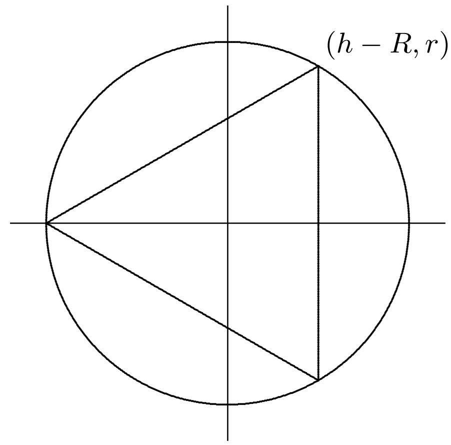


Figure 6.1.4 Cone in a sphere.

figure 6.1.4. We have situated the picture in a convenient way relative to the x and y axes, namely, with the center of the sphere at the origin and the vertex of the cone at the far left on the x-axis.

Notice that the function we want to maximize, $\pi r^2 h/3$, depends on *two* variables. This is frequently the case, but often the two variables are related in some way so that "really" there is only one variable. So our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure: the upper corner of the triangle, whose coordinates are $(h-R, r)$, must be on the circle of radius R. That is,

$$(h-R)^2 + r^2 = R^2.$$

We can solve for h in terms of r or for r in terms of h. Either involves taking a square root, but we notice that the volume function contains r^2, not r by itself, so it is easiest to solve for r^2 directly: $r^2 = R^2 - (h-R)^2$. Then we substitute the result into $\pi r^2 h/3$:

$$\begin{aligned} V(h) &= \pi(R^2 - (h-R)^2)h/3 \\ &= -\frac{\pi}{3}h^3 + \frac{2}{3}\pi h^2 R \end{aligned}$$

We want to maximize $V(h)$ when h is between 0 and $2R$. Now we solve $0 = f'(h) = -\pi h^2 + (4/3)\pi h R$, getting $h = 0$ or $h = 4R/3$. We compute $V(0) = V(2R) = 0$ and $V(4R/3) = (32/81)\pi R^3$. The maximum is the latter; since the volume of the sphere is $(4/3)\pi R^3$, the fraction of the sphere occupied by the cone is

$$\frac{(32/81)\pi R^3}{(4/3)\pi R^3} = \frac{8}{27} \approx 30\%.$$

□

EXAMPLE 6.1.11 You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder. Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.

Let us first choose letters to represent various things: h for the height, r for the base radius, V for the volume of the cylinder, and c for the cost per unit area of the lateral side of the cylinder; V and c are constants, h and r are variables. Now we can write the cost of materials:

$$c(2\pi rh) + Nc(2\pi r^2).$$

Again we have two variables; the relationship is provided by the fixed volume of the cylinder: $V = \pi r^2 h$. We use this relationship to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). The result is

$$f(r) = 2c\pi r \frac{V}{\pi r^2} + 2Nc\pi r^2 = \frac{2cV}{r} + 2Nc\pi r^2.$$

We want to know the minimum value of this function when r is in $(0, \infty)$. We now set $0 = f'(r) = -2cV/r^2 + 4Nc\pi r$, giving $r = \sqrt[3]{V/(2N\pi)}$. Since $f''(r) = 4cV/r^3 + 4Nc\pi$ is positive when r is positive, there is a local minimum at the critical value, and hence a global minimum since there is only one critical value.

Finally, since $h = V/(\pi r^2)$,

$$\frac{h}{r} = \frac{V}{\pi r^3} = \frac{V}{\pi(V/(2N\pi))} = 2N,$$

so the minimum cost occurs when the height h is $2N$ times the radius. If, for example, there is no difference in the cost of materials, the height is twice the radius (or the height is equal to the diameter). □

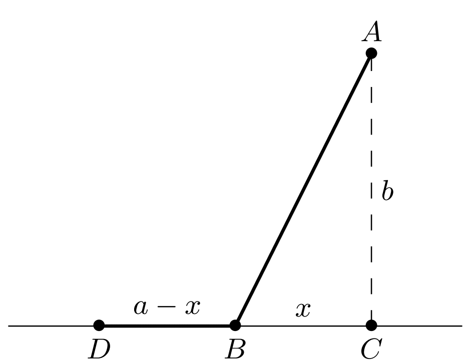


Figure 6.1.5 Minimizing travel time.

EXAMPLE 6.1.12 Suppose you want to reach a point A that is located across the sand from a nearby road (see figure 6.1.5). Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Let x be the distance short of C where you turn off, i.e., the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is distance divided by velocity.

You travel the distance $\overline{DB}$ at speed v, and then the distance $\overline{BA}$ at speed w. Since $\overline{DB} = a - x$ and, by the Pythagorean theorem, $\overline{BA} = \sqrt{x^2 + b^2}$, the total time for the trip is

$$f(x) = \frac{a-x}{v} + \frac{\sqrt{x^2+b^2}}{w}.$$

We want to find the minimum value of f when x is between 0 and a. As usual we set $f'(x) = 0$ and solve for x:

$$\begin{aligned}
0 = f'(x) &= -\frac{1}{v} + \frac{x}{w\sqrt{x^2+b^2}} \\
w\sqrt{x^2+b^2} &= vx \\
w^2(x^2+b^2) &= v^2x^2 \\
w^2b^2 &= (v^2 - w^2)x^2 \\
x &= \frac{wb}{\sqrt{v^2-w^2}}
\end{aligned}$$

Notice that a does not appear in the last expression, but a is not irrelevant, since we are interested only in critical values that are in $[0, a]$, and $wb/\sqrt{v^2 - w^2}$ is either in this interval or not. If it is, we can use the second derivative to test it:

$$f''(x) = \frac{b^2}{(x^2+b^2)^{3/2}w}.$$

Since this is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0, a]$ it is larger than a. In this case the minimum must occur at one of the endpoints. We can compute

$$\begin{aligned}
f(0) &= \frac{a}{v} + \frac{b}{w} \\
f(a) &= \frac{\sqrt{a^2+b^2}}{w}
\end{aligned}$$

but it is difficult to determine which of these is smaller by direct comparison. If, as is likely in practice, we know the values of v, w, a, and b, then it is easy to determine this. With a little cleverness, however, we can determine the minimum in general. We have seen that $f''(x)$ is always positive, so the derivative $f'(x)$ is always increasing. We know that at $wb/\sqrt{v^2-w^2}$ the derivative is zero, so for values of x less than that critical value, the derivative is negative. This means that $f(0) > f(a)$, so the minimum occurs when $x = a$.

So the upshot is this: If you start farther away from C than $wb/\sqrt{v^2-w^2}$ then you always want to cut across the sand when you are a distance $wb/\sqrt{v^2-w^2}$ from point C. If you start closer than this to C, you should cut directly across the sand. □

Summary—Steps to solve an optimization problem.

1. Decide what the variables are and what the constants are, draw a diagram if appropriate, understand clearly what it is that is to be maximized or minimized.
2. Write a formula for the function for which you wish to find the maximum or minimum.
3. Express that formula in terms of only one variable, that is, in the form $f(x)$.
4. Set $f'(x) = 0$ and solve. Check all critical values and endpoints to determine the extreme value.

Exercises 6.1.

1. Let $f(x) = \begin{cases} 1+4x-x^2 & \text{for } x \le 3 \\ (x+5)/2 & \text{for } x > 3 \end{cases}$

Find the maximum value and minimum values of $f(x)$ for x in $[0,4]$. Graph $f(x)$ to check your answers. ⇒

2. Find the dimensions of the rectangle of largest area having fixed perimeter 100. ⇒

3. Find the dimensions of the rectangle of largest area having fixed perimeter P. ⇒

4. A box with square base and no top is to hold a volume 100. Find the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. ⇒

5. A box with square base is to hold a volume 200. The bottom and top are formed by folding in flaps from all four sides, so that the bottom and top consist of two layers of cardboard. Find the dimensions of the box that requires the least material. Also find the ratio of height to side of the base. ⇒

6. A box with square base and no top is to hold a volume V. Find (in terms of V) the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. (This ratio will not involve V.) ⇒

7. You have 100 feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area? ⇒

8. You have l feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area? ⇒

9. Marketing tells you that if you set the price of an item at \$10 then you will be unable to sell it, but that you can sell 500 items for each dollar below \$10 that you set the price. Suppose your fixed costs total \$3000, and your marginal cost is \$2 per item. What is the most profit you can make? ⇒

10. Find the area of the largest rectangle that fits inside a semicircle of radius 10 (one side of the rectangle is along the diameter of the semicircle). ⇒

11. Find the area of the largest rectangle that fits inside a semicircle of radius r (one side of the rectangle is along the diameter of the semicircle). ⇒

12. For a cylinder with surface area 50, including the top and the bottom, find the ratio of height to base radius that maximizes the volume. ⇒

13. For a cylinder with given surface area S, including the top and the bottom, find the ratio of height to base radius that maximizes the volume. ⇒

14. You want to make cylindrical containers to hold 1 liter (1000 cubic centimeters) using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side $2r$, so that $2(2r)^2 = 8r^2$ of material is needed (rather than $2\pi r^2$, which is the total area of the top and bottom). Find the dimensions of the container using the least amount of material, and also find the ratio of height to radius for this container. ⇒

15. You want to make cylindrical containers of a given volume V using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side $2r$, so that $2(2r)^2 = 8r^2$ of material is needed (rather than $2\pi r^2$, which is the total area of the top and bottom). Find the optimal ratio of height to radius. ⇒

16. Given a right circular cone, you put an upside-down cone inside it so that its vertex is at the center of the base of the larger cone and its base is parallel to the base of the larger cone. If you choose the upside-down cone to have the largest possible volume, what fraction of the volume of the larger cone does it occupy? (Let H and R be the height and base radius of the larger cone, and let h and r be the height and base radius of the smaller cone. Hint: Use similar triangles to get an equation relating h and r.) ⇒

17. In example 6.1.12, what happens if $w \geq v$ (i.e., your speed on sand is at least your speed on the road)? ⇒

18. A container holding a fixed volume is being made in the shape of a cylinder with a hemispherical top. (The hemispherical top has the same radius as the cylinder.) Find the ratio of height to radius of the cylinder which minimizes the cost of the container if (a) the cost per unit area of the top is twice as great as the cost per unit area of the side, and the container is made with no bottom; (b) the same as in (a), except that the container is made with a circular bottom, for which the cost per unit area is 1.5 times the cost per unit area of the side. ⇒

19. A piece of cardboard is 1 meter by 1/2 meter. A square is to be cut from each corner and the sides folded up to make an open-top box. What are the dimensions of the box with maximum possible volume? ⇒

20. (a) A square piece of cardboard of side a is used to make an open-top box by cutting out a small square from each corner and bending up the sides. How large a square should be cut from each corner in order that the box have maximum volume? (b) What if the piece of cardboard used to make the box is a rectangle of sides a and b? ⇒

21. A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top; the colored glass transmits only 1/2 as much light per unit area as the the clear glass. If the distance from top to bottom (across both the rectangle and the semicircle) is 2 meters and the window may be no more than 1.5 meters wide, find the dimensions of the rectangular portion of the window that lets through the most light. ⇒

22. A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top. Suppose that the colored glass transmits only k times as much light per unit area as the clear glass (k is between 0 and 1). If the distance from top to bottom (across both the rectangle and the semicircle) is a fixed distance H, find (in terms of k) the ratio of vertical side to horizontal side of the rectangle for which the window lets through the most light. ⇒

23. You are designing a poster to contain a fixed amount A of printing (measured in square centimeters) and have margins of a centimeters at the top and bottom and b centimeters at the sides. Find the ratio of vertical dimension to horizontal dimension of the printed area on the poster if you want to minimize the amount of posterboard needed. ⇒

24. The strength of a rectangular beam is proportional to the product of its width w times the square of its depth d. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius r. ⇒

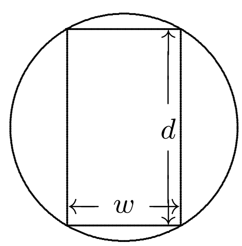


Figure 6.1.6 Cutting a beam.

25. What fraction of the volume of a sphere is taken up by the largest cylinder that can be fit inside the sphere? ⇒

26. The U.S. post office will accept a box for shipment only if the sum of the length and girth (distance around) is at most 108 in. Find the dimensions of the largest acceptable box with square front and back. ⇒

27. Find the dimensions of the lightest cylindrical can containing 0.25 liter (=250 cm^3) if the top and bottom are made of a material that is twice as heavy (per unit area) as the material used for the side. ⇒

28. A conical paper cup is to hold 1/4 of a liter. Find the height and radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula $\pi r\sqrt{r^2+h^2}$ for the area of the side of a cone. ⇒

29. A conical paper cup is to hold a fixed volume of water. Find the ratio of height to base radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula $\pi r\sqrt{r^2+h^2}$ for the area of the side of a cone, called the **lateral area** of the cone. ⇒

30. If you fit the cone with the largest possible surface area (lateral area plus area of base) into a sphere, what percent of the volume of the sphere is occupied by the cone? ⇒

31. Two electrical charges, one a positive charge A of magnitude a and the other a negative charge B of magnitude b, are located a distance c apart. A positively charged particle P is situated on the line between A and B. Find where P should be put so that the pull away from A towards B is minimal. Here assume that the force from each charge is proportional to the strength of the source and inversely proportional to the square of the distance from the source. ⇒

32. Find the fraction of the area of a triangle that is occupied by the largest rectangle that can be drawn in the triangle (with one of its sides along a side of the triangle). Show that this fraction does not depend on the dimensions of the given triangle. ⇒

33. How are your answers to Problem 9 affected if the cost per item for the x items, instead of being simply \$2, decreases below \$2 in proportion to x (because of economy of scale and volume discounts) by 1 cent for each 25 items produced? ⇒

34. You are standing near the side of a large wading pool of uniform depth when you see a child in trouble. You can run at a speed v_1 on land and at a slower speed v_2 in the water. Your perpendicular distance from the side of the pool is a, the child's perpendicular distance is b, and the distance along the side of the pool between the closest point to you and the closest point to the child is c (see the figure below). Without stopping to do any calculus, you instinctively choose the quickest route (shown in the figure) and save the child. Our purpose is to derive a relation between the angle θ_1 your path makes with the perpendicular to the side of the pool when you're on land, and the angle θ_2 your path makes with the perpendicular when you're in the water. To do this, let x be the distance between the closest point to you at the side of the pool and the point where you enter the water. Write the total time you run (on land and in the water) in terms of x (and also the constants a, b, c, v_1, v_2). Then set the derivative equal to zero. The result, called "Snell's law" or the "law of refraction," also governs the bending of light when it goes into water. ⇒

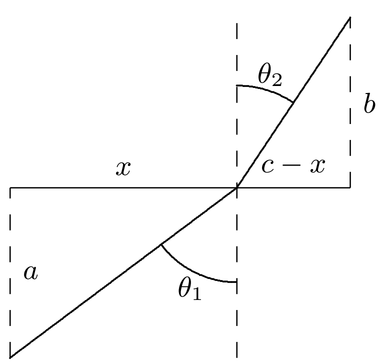


Figure 6.1.7 Wading pool rescue.

6.2 Related Rates

Suppose we have two variables x and y (in most problems the letters will be different, but for now let's use x and y) which are both changing with time. A "related rates" problem is a problem in which we know one of the rates of change at a given instant—say,

$\dot{x} = dx/dt$—and we want to find the other rate $\dot{y} = dy/dt$ at that instant. (The use of $\dot{x}$ to mean dx/dt goes back to Newton and is still used for this purpose, especially by physicists.)

If y is written in terms of x, i.e., $y = f(x)$, then this is easy to do using the chain rule:

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx}\dot{x}.$$

That is, find the derivative of $f(x)$, plug in the value of x at the instant in question, and multiply by the given value of $\dot{x} = dx/dt$ to get $\dot{y} = dy/dt$.

EXAMPLE 6.2.1 Suppose an object is moving along a path described by $y = x^2$, that is, it is moving on a parabolic path. At a particular time, say $t = 5$, the x coordinate is 6 and we measure the speed at which the x coordinate of the object is changing and find that $dx/dt = 3$. At the same time, how fast is the y coordinate changing?

Using the chain rule, $dy/dt = 2x \cdot dx/dt$. At $t = 5$ we know that $x = 6$ and $dx/dt = 3$, so $dy/dt = 2 \cdot 6 \cdot 3 = 36$. □

In many cases, particularly interesting ones, x and y will be related in some other way, for example $x = f(y)$, or $F(x, y) = k$, or perhaps $F(x, y) = G(x, y)$, where $F(x, y)$ and $G(x, y)$ are expressions involving both variables. In all cases, you can solve the related rates problem by taking the derivative of both sides, plugging in all the known values (namely, x, y, and $\dot{x}$), and then solving for $\dot{y}$.

To summarize, here are the steps in doing a related rates problem:

1. Decide what the two variables are.

2. Find an equation relating them.

3. Take d/dt of both sides.

4. Plug in all known values at the instant in question.

5. Solve for the unknown rate.

EXAMPLE 6.2.2 A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

To see what's going on, we first draw a schematic representation of the situation, as in figure 6.2.1.

Because the plane is in level flight directly away from you, the rate at which x changes is the speed of the plane, $dx/dt = 500$. The distance between you and the plane is y; it is dy/dt that we wish to know. By the Pythagorean Theorem we know that $x^2 + 9 = y^2$.

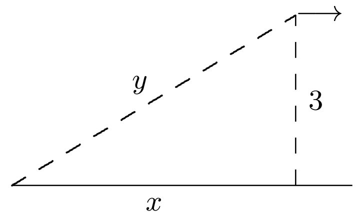


Figure 6.2.1 Receding airplane.

Taking the derivative:

$$2x\dot{x} = 2y\dot{y}.$$

We are interested in the time at which $x = 4$; at this time we know that $4^2 + 9 = y^2$, so $y = 5$. Putting together all the information we get

$$2(4)(500) = 2(5)\dot{y}.$$

Thus, $\dot{y} = 400$ mph. □

EXAMPLE 6.2.3 You are inflating a spherical balloon at the rate of 7 cm^3/sec. How fast is its radius increasing when the radius is 4 cm?

Here the variables are the radius r and the volume V. We know dV/dt, and we want dr/dt. The two variables are related by means of the equation $V = 4\pi r^3/3$. Taking the derivative of both sides gives $dV/dt = 4\pi r^2 \dot{r}$. We now substitute the values we know at the instant in question: $7 = 4\pi 4^2 \dot{r}$, so $\dot{r} = 7/(64\pi)$ cm/sec. □

EXAMPLE 6.2.4 Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm; see figure 6.2.2. How fast is the water level rising when the water is 4 cm deep (at its deepest point)?

The water forms a conical shape within the big cone; its height and base radius and volume are all increasing as water is poured into the container. This means that we actually have three things varying with time: the water level h (the height of the cone of water), the radius r of the circular top surface of water (the base radius of the cone of water), and the volume of water V. The volume of a cone is given by $V = \pi r^2 h/3$. We know dV/dt, and we want dh/dt. At first something seems to be wrong: we have a third variable r whose rate we don't know.

But the dimensions of the cone of water must have the same proportions as those of the container. That is, because of similar triangles, $r/h = 10/30$ so $r = h/3$. Now we can eliminate r from the problem entirely: $V = \pi(h/3)^2 h/3 = \pi h^3/27$. We take the derivative of both sides and plug in $h = 4$ and $dV/dt = 10$, obtaining $10 = (3\pi \cdot 4^2/27)(dh/dt)$. Thus, $dh/dt = 90/(16\pi)$ cm/sec. □

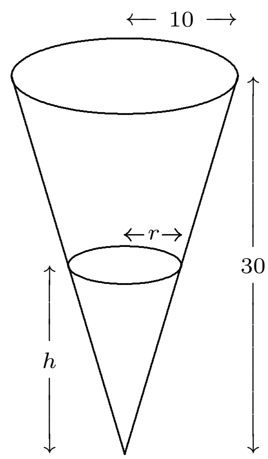


Figure 6.2.2 Conical water tank.

EXAMPLE 6.2.5 A swing consists of a board at the end of a 10 ft long rope. Think of the board as a point P at the end of the rope, and let Q be the point of attachment at the other end. Suppose that the swing is directly below Q at time $t = 0$, and is being pushed by someone who walks at 6 ft/sec from left to right. Find (a) how fast the swing is rising after 1 sec; (b) the angular speed of the rope in deg/sec after 1 sec.

We start out by asking: What is the geometric quantity whose rate of change we know, and what is the geometric quantity whose rate of change we're being asked about? Note that the person pushing the swing is moving horizontally at a rate we know. In other words, the horizontal coordinate of P is increasing at 6 ft/sec. In the xy-plane let us make the convenient choice of putting the origin at the location of P at time $t = 0$, i.e., a distance 10 directly below the point of attachment. Then the rate we know is dx/dt, and in part (a) the rate we want is dy/dt (the rate at which P is rising). In part (b) the rate we want is $\dot{\theta} = d\theta/dt$, where θ stands for the angle in radians through which the swing has swung from the vertical. (Actually, since we want our answer in deg/sec, at the end we must convert $d\theta/dt$ from rad/sec by multiplying by $180/\pi$.)

(a) From the diagram we see that we have a right triangle whose legs are x and $10 - y$, and whose hypotenuse is 10. Hence $x^2 + (10 - y)^2 = 100$. Taking the derivative of both sides we obtain: $2x\dot{x} + 2(10 - y)(0 - \dot{y}) = 0$. We now look at what we know after 1 second, namely $x = 6$ (because x started at 0 and has been increasing at the rate of 6 ft/sec for 1 sec), $y = 2$ (because we get $10 - y = 8$ from the Pythagorean theorem applied to the triangle with hypotenuse 10 and leg 6), and $\dot{x} = 6$. Putting in these values gives us $2 \cdot 6 \cdot 6 - 2 \cdot 8\dot{y} = 0$, from which we can easily solve for $\dot{y}$: $\dot{y} = 4.5$ ft/sec.

(b) Here our two variables are x and θ, so we want to use the same right triangle as in part (a), but this time relate θ to x. Since the hypotenuse is constant (equal to 10), the best way to do this is to use the sine: $\sin\theta = x/10$. Taking derivatives we obtain

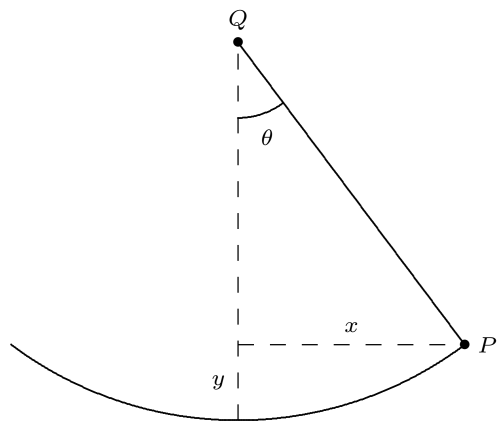


Figure 6.2.3 Swing.

$(\cos\theta)\dot{\theta} = 0.1\dot{x}$. At the instant in question ($t = 1$ sec), when we have a right triangle with sides 6–8–10, $\cos\theta = 8/10$ and $\dot{x} = 6$. Thus $(8/10)\dot{\theta} = 6/10$, i.e., $\dot{\theta} = 6/8 = 3/4$ rad/sec, or approximately 43 deg/sec. □

We have seen that sometimes there are apparently more than two variables that change with time, but in reality there are just two, as the others can be expressed in terms of just two. But sometimes there really are several variables that change with time; as long as you know the rates of change of all but one of them you can find the rate of change of the remaining one. As in the case when there are just two variables, take the derivative of both sides of the equation relating all of the variables, and then substitute all of the known values and solve for the unknown rate.

EXAMPLE 6.2.6 A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometers to the north of P and traveling at 80 km/hr, while car B is 15 kilometers to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

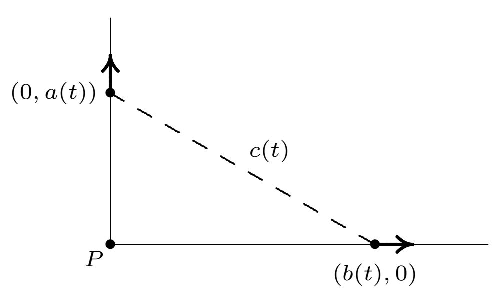


Figure 6.2.4 Cars moving apart.

Let $a(t)$ be the distance of car A north of P at time t, and $b(t)$ the distance of car B east of P at time t, and let $c(t)$ be the distance from car A to car B at time t. By the Pythagorean Theorem, $c(t)^2 = a(t)^2 + b(t)^2$. Taking derivatives we get $2c(t)c'(t) = 2a(t)a'(t) + 2b(t)b'(t)$, so

$$\dot{c} = \frac{a\dot{a} + b\dot{b}}{c} = \frac{a\dot{a} + b\dot{b}}{\sqrt{a^2 + b^2}}.$$

Substituting known values we get:

$$\dot{c} = \frac{10 \cdot 80 + 15 \cdot 100}{\sqrt{10^2 + 15^2}} = \frac{460}{\sqrt{13}} \approx 127.6\text{km/hr}$$

at the time of interest. □

Notice how this problem differs from example 6.2.2. In both cases we started with the Pythagorean Theorem and took derivatives on both sides. However, in example 6.2.2 one of the sides was a constant (the altitude of the plane), and so the derivative of the square of that side of the triangle was simply zero. In this example, on the other hand, all three sides of the right triangle are variables, even though we are interested in a specific value of each side of the triangle (namely, when the sides have lengths 10 and 15). Make sure that you understand at the start of the problem what are the variables and what are the constants.

Exercises 6.2.

1. A cylindrical tank standing upright (with one circular base on the ground) has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 25 cm^3/sec? ⇒

2. A cylindrical tank standing upright (with one circular base on the ground) has radius 1 meter. How fast does the water level in the tank drop when the water is being drained at 3 liters per second? ⇒

3. A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall? ⇒

4. A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The top of the ladder is being pulled up the wall at 0.1 meters per second. How fast is the foot of the ladder approaching the wall when the foot of the ladder is 5 m from the wall? ⇒

5. A rotating beacon is located 2 miles out in the water. Let A be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point where the beam hits the shore moving at an instant when the beam is lighting up a point 2 miles along the shore from the point A? ⇒

6. A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 15 ft/sec. At what rate is the player's distance from third base decreasing when she is half way from first to second base? ⇒

7. Sand is poured onto a surface at 15 cm^3/sec, forming a conical pile whose base diameter is always equal to its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high? ⇒

8. A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 5 ft higher than the front of the boat. The rope is being pulled through the ring at the rate of 0.6 ft/sec. How fast is the boat approaching the dock when 13 ft of rope are out? ⇒

9. A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. A bicyclist passes beneath it, traveling in a straight line at the constant speed of 10 m/sec. How fast is the distance between the bicyclist and the balloon increasing 2 seconds later? ⇒

10. A pyramid-shaped vat has square cross-section and stands on its tip. The dimensions at the top are 2 m × 2 m, and the depth is 5 m. If water is flowing into the vat at 3 m^3/min, how fast is the water level rising when the depth of water (at the deepest point) is 4 m? Note: the volume of any "conical" shape (including pyramids) is (1/3)(height)(area of base). ⇒

11. The sun is rising at the rate of 1/4 deg/min, and appears to be climbing into the sky perpendicular to the horizon, as depicted in figure 6.2.5. How fast is the shadow of a 200 meter building shrinking at the moment when the shadow is 500 meters long? ⇒

12. The sun is setting at the rate of 1/4 deg/min, and appears to be dropping perpendicular to the horizon, as depicted in figure 6.2.5. How fast is the shadow of a 25 meter wall lengthening at the moment when the shadow is 50 meters long? ⇒

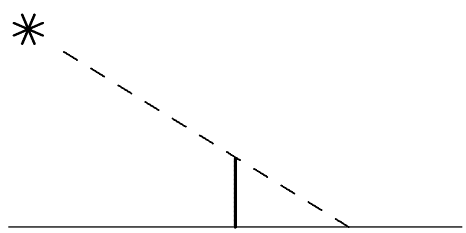


Figure 6.2.5 Sunrise or sunset.

13. The trough shown in figure 6.2.6 is constructed by fastening together three slabs of wood of dimensions 10 ft × 1 ft, and then attaching the construction to a wooden wall at each end. The angle θ was originally 30°, but because of poor construction the sides are collapsing. The trough is full of water. At what rate (in ft^3/sec) is the water spilling out over the top of the trough if the sides have each fallen to an angle of 45°, and are collapsing at the rate of 1° per second? ⇒

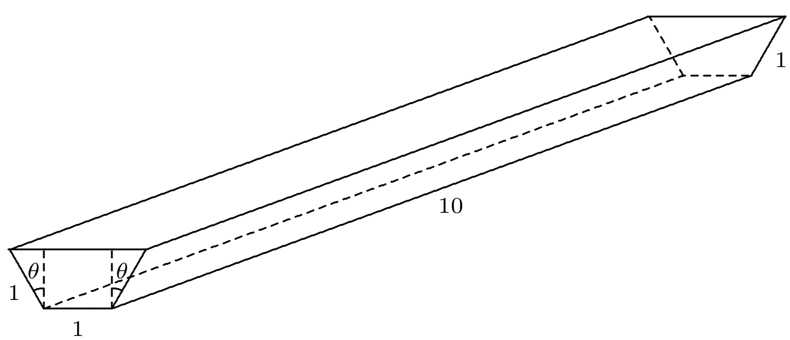


Figure 6.2.6 Trough.

14. A woman 5 ft tall walks at the rate of 3.5 ft/sec away from a streetlight that is 12 ft above the ground. At what rate is the tip of her shadow moving? At what rate is her shadow lengthening? ⇒

15. A man 1.8 meters tall walks at the rate of 1 meter per second toward a streetlight that is 4 meters above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow shortening? ⇒

16. A police helicopter is flying at 150 mph at a constant altitude of 0.5 mile above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 mile from the helicopter, and that this distance is decreasing at 190 mph. Find the speed of the car. ⇒

17. A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 2 kilometers from the helicopter, and that this distance is decreasing at 250 kph. Find the speed of the car. ⇒

18. A light shines from the top of a pole 20 m high. A ball is falling 10 meters from the pole, casting a shadow on a building 30 meters away, as shown in figure 6.2.7. When the ball is 25 meters from the ground it is falling at 6 meters per second. How fast is its shadow moving? ⇒

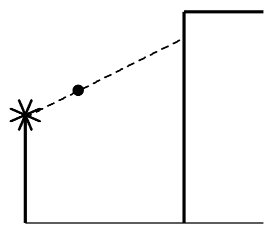


Figure 6.2.7 Falling ball.

19. Do example 6.2.6 assuming that the angle between the two roads is 120° instead of 90° (that is, the "north–south" road actually goes in a somewhat northwesterly direction from P). Recall the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$. ⇒

20. Do example 6.2.6 assuming that car A is 300 meters north of P, car B is 400 meters east of P, both cars are going at constant speed toward P, and the two cars will collide in 10 seconds. ⇒

21. Do example 6.2.6 assuming that 8 seconds ago car A started from rest at P and has been picking up speed at the steady rate of 5 m/sec^2, and 6 seconds after car A started car B passed P moving east at constant speed 60 m/sec. ⇒

22. Referring again to example 6.2.6, suppose that instead of car B an airplane is flying at speed 200 km/hr to the east of P at an altitude of 2 km, as depicted in figure 6.2.8. How fast is the distance between car and airplane changing? ⇒

23. Referring again to example 6.2.6, suppose that instead of car B an airplane is flying at speed 200 km/hr to the east of P at an altitude of 2 km, and that it is gaining altitude at 10 km/hr. How fast is the distance between car and airplane changing? ⇒

24. A light shines from the top of a pole 20 m high. An object is dropped from the same height from a point 10 m away, so that its height at time t seconds is $h(t) = 20 - 9.8t^2/2$. How fast is the object's shadow moving on the ground one second later? ⇒

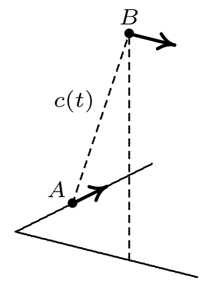


Figure 6.2.8 Car and airplane.

25. The two blades of a pair of scissors are fastened at the point A as shown in figure 6.2.9. Let a denote the distance from A to the tip of the blade (the point B). Let β denote the angle at the tip of the blade that is formed by the line $\overline{AB}$ and the bottom edge of the blade, line $\overline{BC}$, and let θ denote the angle between $\overline{AB}$ and the horizontal. Suppose that a piece of paper is cut in such a way that the center of the scissors at A is fixed, and the paper is also fixed. As the blades are closed (i.e., the angle θ in the diagram is decreased), the distance x between A and C increases, cutting the paper.

- **a.** Express x in terms of a, θ, and β.
- **b.** Express dx/dt in terms of a, θ, β, and $d\theta/dt$.
- **c.** Suppose that the distance a is 20 cm, and the angle β is 5°. Further suppose that θ is decreasing at 50 deg/sec. At the instant when $\theta = 30^\circ$, find the rate (in cm/sec) at which the paper is being cut. ⇒

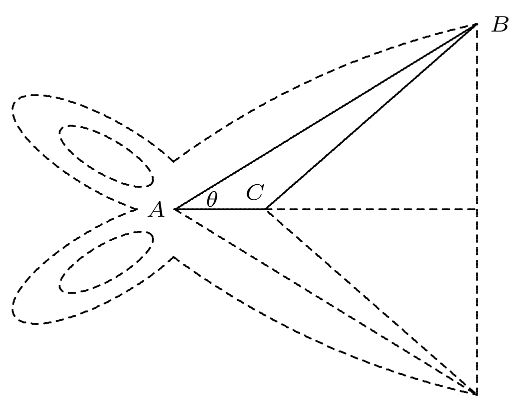


Figure 6.2.9 Scissors.

6.3 Newton's Method

Suppose you have a function $f(x)$, and you want to find as accurately as possible where it crosses the x-axis; in other words, you want to solve $f(x) = 0$. Suppose you know of no way to find an exact solution by any algebraic procedure, but you are able to use an approximation, provided it can be made quite close to the true value. Newton's method is a way to find a solution to the equation to as many decimal places as you want. It is what is called an "iterative procedure," meaning that it can be repeated again and again to get an answer of greater and greater accuracy. Iterative procedures like Newton's method are well suited to programming for a computer. Newton's method uses the fact that the tangent line to a curve is a good approximation to the curve near the point of tangency.

EXAMPLE 6.3.1 Approximate $\sqrt{3}$. Since $\sqrt{3}$ is a solution to $x^2 = 3$ or $x^2 - 3 = 0$, we use $f(x) = x^2 - 3$. We start by guessing something reasonably close to the true value; this is usually easy to do; let's use $\sqrt{3} \approx 2$. Now use the tangent line to the curve when $x = 2$ as an approximation to the curve, as shown in figure 6.3.1. Since $f'(x) = 2x$, the slope of this tangent line is 4 and its equation is $y = 4x - 7$. The tangent line is quite close to $f(x)$, so it crosses the x-axis near the point at which $f(x)$ crosses, that is, near $\sqrt{3}$. It is easy to find where the tangent line crosses the x-axis: solve $0 = 4x - 7$ to get $x = 7/4 = 1.75$. This is certainly a better approximation than 2, but let us say not close enough. We can improve it by doing the same thing again: find the tangent line at $x = 1.75$, find where this new tangent line crosses the x-axis, and use that value as a better approximation. We can continue this indefinitely, though it gets a bit tedious. Lets see if we can shortcut the process. Suppose the best approximation to the intercept we have so far is x_i. To find a better approximation we will always do the same thing: find the slope of the tangent line at x_i, find the equation of the tangent line, find the x-intercept. The slope is $2x_i$. The tangent line is $y = (2x_i)(x - x_i) + (x_i^2 - 3)$, using the point-slope formula for a line. Finally, the intercept is found by solving $0 = (2x_i)(x - x_i) + (x_i^2 - 3)$. With a little algebra this turns into $x = (x_i^2 + 3)/(2x_i)$; this is the next approximation, which we naturally call x_{i+1}. Instead of doing the whole tangent line computation every time we can simply use this formula to get as many approximations as we want. Starting with $x_0 = 2$, we get $x_1 = (x_0^2 + 3)/(2x_0) = (2^2 + 3)/4 = 7/4$ (the same approximation we got above, of course), $x_2 = (x_1^2 + 3)/(2x_1) = ((7/4)^2 + 3)/(7/2) = 97/56 \approx 1.73214$, $x_3 \approx 1.73205$, and so on. This is still a bit tedious by hand, but with a calculator or, even better, a good computer program, it is quite easy to get many, many approximations. We might guess already that 1.73205 is accurate to two decimal places, and in fact it turns out that it is accurate to 5 places. □

Let's think about this process in more general terms. We want to approximate a solution to $f(x) = 0$. We start with a rough guess, which we call x_0. We use the tangent line to $f(x)$ to get a new approximation that we hope will be closer to the true value. What is the equation of the tangent line when $x = x_0$? The slope is $f'(x_0)$ and the line goes through $(x_0, f(x_0))$, so the equation of the line is

$$y = f'(x_0)(x - x_0) + f(x_0).$$

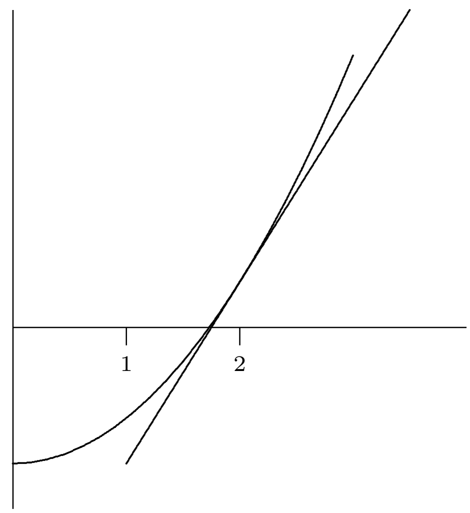


Figure 6.3.1 Newton's method. (AP)

Now we find where this crosses the x-axis by substituting $y = 0$ and solving for x:

$$x = \frac{x_0 f'(x_0) - f(x_0)}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

We will typically want to compute more than one of these improved approximations, so we number them consecutively; from x_0 we have computed x_1:

$$x_1 = \frac{x_0 f'(x_0) - f(x_0)}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)},$$

and in general from x_i we compute x_{i+1}:

$$x_{i+1} = \frac{x_i f'(x_i) - f(x_i)}{f'(x_i)} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

EXAMPLE 6.3.2 Returning to the previous example, $f(x) = x^2 - 3$, $f'(x) = 2x$, and the formula becomes $x_{i+1} = x_i - (x_i^2 - 3)/(2x_i) = (x_i^2 + 3)/(2x_i)$, as before. □

In practice, which is to say, if you need to approximate a value in the course of designing a bridge or a building or an airframe, you will need to have some confidence that the approximation you settle on is accurate enough. As a rule of thumb, once a certain number of decimal places stop changing from one approximation to the next it is likely that those decimal places are correct. Still, this may not be enough assurance, in which case we can test the result for accuracy.

EXAMPLE 6.3.3 Find the x coordinate of the intersection of the curves $y = 2x$ and $y = \tan x$, accurate to three decimal places. To put this in the context of Newton's method, we note that we want to know where $2x = \tan x$ or $f(x) = \tan x - 2x = 0$. We compute $f'(x) = \sec^2 x - 2$ and set up the formula:

$$x_{i+1} = x_i - \frac{\tan x_i - 2x_i}{\sec^2 x_i - 2}.$$

From the graph in figure 6.3.2 we guess $x_0 = 1$ as a starting point, then using the formula we compute $x_1 = 1.310478030$, $x_2 = 1.223929096$, $x_3 = 1.176050900$, $x_4 = 1.165926508$, $x_5 = 1.165561636$. So we guess that the first three places are correct, but that is not the same as saying 1.165 is correct to three decimal places—1.166 might be the correct, rounded approximation. How can we tell? We can substitute 1.165, 1.1655 and 1.166 into $\tan x - 2x$; this gives −0.002483652, −0.000271247, 0.001948654. Since the first two are negative and the third is positive, $\tan x - 2x$ crosses the x axis between 1.1655 and 1.166, so the correct value to three places is 1.166. □

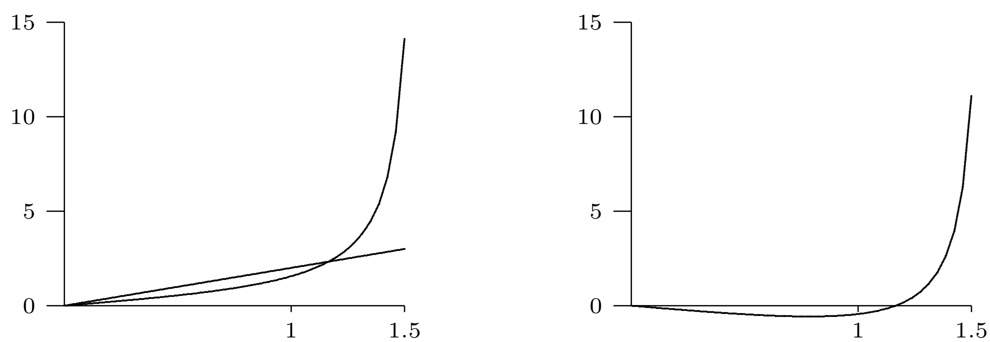


Figure 6.3.2 $y = \tan x$ and $y = 2x$ on the left, $y = \tan x - 2x$ on the right.

Exercises 6.3.

1. Approximate the fifth root of 7, using $x_0 = 1.5$ as a first guess. Use Newton's method to find x_3 as your approximation. ⇒
2. Use Newton's Method to approximate the cube root of 10 to two decimal places. ⇒
3. The function $f(x) = x^3 - 3x^2 - 3x + 6$ has a root between 3 and 4, because $f(3) = -3$ and $f(4) = 10$. Approximate the root to two decimal places. ⇒
4. A rectangular piece of cardboard of dimensions 8×17 is used to make an open-top box by cutting out a small square of side x from each corner and bending up the sides. (See exercise 20 in 6.1.) If $x = 2$, then the volume of the box is $2 \cdot 4 \cdot 13 = 104$. Use Newton's method to find a value of x for which the box has volume 100, accurate to 3 significant figures. ⇒

6.4 Linear Approximations

Newton's method is one example of the usefulness of the tangent line as an approximation to a curve. Here we explore another such application.

Recall that the tangent line to $f(x)$ at a point $x = a$ is given by $L(x) = f'(a)(x - a) + f(a)$. The tangent line in this context is also called the **linear approximation** to f at a.

If f is differentiable at a then L is a good approximation of f so long as x is "not too far" from a. Put another way, if f is differentiable at a then under a microscope f will look very much like a straight line. Figure 6.4.1 shows a tangent line to $y = x^2$ at three different magnifications.

If we want to approximate $f(b)$, because computing it exactly is difficult, we can approximate the value using a linear approximation, provided that we can compute the tangent line at some a close to b.

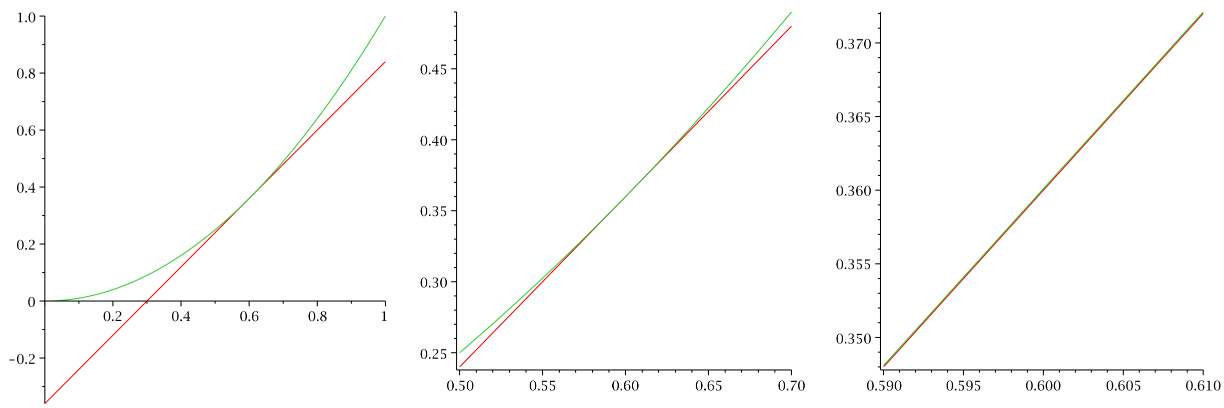


Figure 6.4.1 The linear approximation to $y = x^2$.

EXAMPLE 6.4.1 Let $f(x) = \sqrt{x+4}$. Then $f'(x) = 1/(2\sqrt{x+4})$. The linear approximation to f at $x = 5$ is $L(x) = 1/(2\sqrt{5+4})(x-5) + \sqrt{5+4} = (x-5)/6 + 3$. As an immediate application we can approximate square roots of numbers near 9 by hand. To estimate $\sqrt{10}$, we substitute 6 into the linear approximation instead of into $f(x)$, so $\sqrt{6+4} \approx (6-5)/6 + 3 = 19/6 \approx 3.1\overline{6}$. This rounds to 3.17 while the square root of 10 is actually 3.16 to two decimal places, so this estimate is only accurate to one decimal place. This is not too surprising, as 10 is really not very close to 9; on the other hand, for many calculations, 3.2 would be accurate enough. $\square$

With modern calculators and computing software it may not appear necessary to use linear approximations. But in fact they are quite useful. In cases requiring an explicit numerical approximation, they allow us to get a quick rough estimate which can be used as a "reality check" on a more complex calculation. In some complex calculations involving functions, the linear approximation makes an otherwise intractable calculation possible, without serious loss of accuracy.

EXAMPLE 6.4.2 Consider the trigonometric function $\sin x$. Its linear approximation at $x = 0$ is simply $L(x) = x$. When x is small this is quite a good approximation and is used frequently by engineers and scientists to simplify some calculations. □

DEFINITION 6.4.3 Let $y = f(x)$ be a differentiable function. We define a new independent variable dx, and a new dependent variable $dy = f'(x)\,dx$. Notice that dy is a function both of x (since $f'(x)$ is a function of x) and of dx. We say that dx and dy are **differentials.** □

Let $\Delta x = x - a$ and $\Delta y = f(x) - f(a)$. If x is near a then Δx is small. If we set $dx = \Delta x$ then

$$dy = f'(a)\,dx \approx \frac{\Delta y}{\Delta x}\Delta x = \Delta y.$$

Thus, dy can be used to approximate Δy, the actual change in the function f between a and x. This is exactly the approximation given by the tangent line:

$$dy = f'(a)(x - a) = f'(a)(x - a) + f(a) - f(a) = L(x) - f(a).$$

While $L(x)$ approximates $f(x)$, dy approximates how $f(x)$ has changed from $f(a)$. Figure 6.4.2 illustrates the relationships.

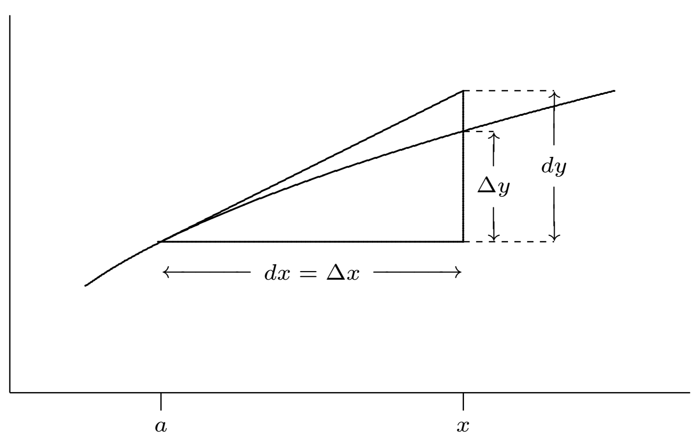


Figure 6.4.2 Differentials.

Exercises 6.4.

1. Let $f(x) = x^4$. If $a = 1$ and $dx = \Delta x = 1/2$, what are Δy and dy? ⇒
2. Let $f(x) = \sqrt{x}$. If $a = 1$ and $dx = \Delta x = 1/10$, what are Δy and dy? ⇒
3. Let $f(x) = \sin(2x)$. If $a = \pi$ and $dx = \Delta x = \pi/100$, what are Δy and dy? ⇒
4. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.02 cm thick to a sphere with diameter 40 meters. (Recall that the volume of a sphere of radius r is $V = (4/3)\pi r^3$. Notice that you are given that $dr = 0.02$.) ⇒
5. Show in detail that the linear approximation of $\sin x$ at $x = 0$ is $L(x) = x$ and the linear approximation of $\cos x$ at $x = 0$ is $L(x) = 1$.

6.5 The Mean Value Theorem

Here are two interesting questions involving derivatives:

1. Suppose two different functions have the same derivative; what can you say about the relationship between the two functions?
2. Suppose you drive a car from toll booth on a toll road to another toll booth at an average speed of 70 miles per hour. What can be concluded about your actual speed during the trip? In particular, did you exceed the 65 mile per hour speed limit?

While these sound very different, it turns out that the two problems are very closely related. We know that "speed" is really the derivative by a different name; let's start by translating the second question into something that may be easier to visualize. Suppose that the function $f(t)$ gives the position of your car on the toll road at time t. Your change in position between one toll booth and the next is given by $f(t_1) - f(t_0)$, assuming that at time t_0 you were at the first booth and at time t_1 you arrived at the second booth. Your average speed for the trip is $(f(t_1) - f(t_0))/(t_1 - t_0)$. If we think about the graph of $f(t)$, the average speed is the slope of the line that connects the two points $(t_0, f(t_0))$ and $(t_1, f(t_1))$. Your speed at any particular time t between t_0 and t_1 is $f'(t)$, the slope of the curve. Now question (2) becomes a question about slope. In particular, if the slope between endpoints is 70, what can be said of the slopes at points between the endpoints?

As a general rule, when faced with a new problem it is often a good idea to examine one or more simplified versions of the problem, in the hope that this will lead to an understanding of the original problem. In this case, the problem in its "slope" form is somewhat easier to simplify than the original, but equivalent, problem.

Here is a special instance of the problem. Suppose that $f(t_0) = f(t_1)$. Then the two endpoints have the same height and the slope of the line connecting the endpoints is zero. What can we say about the slope between the endpoints? It shouldn't take much experimentation before you are convinced of the truth of this statement: Somewhere between t_0 and t_1 the slope is exactly zero, that is, somewhere between t_0 and t_1 the slope is equal to the slope of the line between the endpoints. This suggests that perhaps the same is true even if the endpoints are at different heights, and again a bit of experimentation will probably convince you that this is so. But we can do better than "experimentation"—we can prove that this is so.

We start with the simplified version:

THEOREM 6.5.1 Rolle's Theorem Suppose that $f(x)$ has a derivative on the interval (a, b), is continuous on the interval $[a, b]$, and $f(a) = f(b)$. Then at some value $c \in (a, b)$, $f'(c) = 0$.

Proof. We know that $f(x)$ has a maximum and minimum value on $[a, b]$ (because it is continuous), and we also know that the maximum and minimum must occur at an endpoint, at a point at which the derivative is zero, or at a point where the derivative is undefined. Since the derivative is never undefined, that possibility is removed.

If the maximum or minimum occurs at a point c, other than an endpoint, where $f'(c) = 0$, then we have found the point we seek. Otherwise, the maximum and minimum both occur at an endpoint, and since the endpoints have the same height, the maximum and minimum are the same. This means that $f(x) = f(a) = f(b)$ at every $x \in [a, b]$, so the function is a horizontal line, and it has derivative zero everywhere in (a, b). Then we may choose any c at all to get $f'(c) = 0$. ■

Perhaps remarkably, this special case is all we need to prove the more general one as well.

THEOREM 6.5.2 Mean Value Theorem Suppose that $f(x)$ has a derivative on the interval (a, b) and is continuous on the interval $[a, b]$. Then at some value $c \in (a, b)$, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof. Let $m = \dfrac{f(b) - f(a)}{b - a}$, and consider a new function $g(x) = f(x) - m(x - a) - f(a)$. We know that $g(x)$ has a derivative everywhere, since $g'(x) = f'(x) - m$. We can compute $g(a) = f(a) - m(a - a) - f(a) = 0$ and

$$\begin{aligned} g(b) = f(b) - m(b - a) - f(a) &= f(b) - \frac{f(b) - f(a)}{b - a}(b - a) - f(a) \\ &= f(b) - (f(b) - f(a)) - f(a) = 0. \end{aligned}$$

So the height of $g(x)$ is the same at both endpoints. This means, by Rolle's Theorem, that at some c, $g'(c) = 0$. But we know that $g'(c) = f'(c) - m$, so

$$0 = f'(c) - m = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which turns into

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

exactly what we want. ■

Returning to the original formulation of question (2), we see that if $f(t)$ gives the position of your car at time t, then the Mean Value Theorem says that at some time c, $f'(c) = 70$, that is, at some time you must have been traveling at exactly your average speed for the trip, and that indeed you exceeded the speed limit.

Now let's return to question (1). Suppose, for example, that two functions are known to have derivative equal to 5 everywhere, $f'(x) = g'(x) = 5$. It is easy to find such functions: $5x$, $5x + 47$, $5x - 132$, etc. Are there other, more complicated, examples? No—the only functions that work are the "obvious" ones, namely, $5x$ plus some constant. How can we see that this is true?

Although "5" is a very simple derivative, let's look at an even simpler one. Suppose that $f'(x) = g'(x) = 0$. Again we can find examples: $f(x) = 0$, $f(x) = 47$, $f(x) = -511$ all have $f'(x) = 0$. Are there non-constant functions f with derivative 0? No, and here's why: Suppose that $f(x)$ is not a constant function. This means that there are two points on the function with different heights, say $f(a) \neq f(b)$. The Mean Value Theorem tells us that at some point c, $f'(c) = (f(b) - f(a))/(b - a) \neq 0$. So any non-constant function does not have a derivative that is zero everywhere; this is the same as saying that the only functions with zero derivative are the constant functions.

Let's go back to the slightly less easy example: suppose that $f'(x) = g'(x) = 5$. Then $(f(x) - g(x))' = f'(x) - g'(x) = 5 - 5 = 0$. So using what we discovered in the previous paragraph, we know that $f(x) - g(x) = k$, for some constant k. So any two functions with derivative 5 must differ by a constant; since $5x$ is known to work, the only other examples must look like $5x + k$.

Now we can extend this to more complicated functions, without any extra work. Suppose that $f'(x) = g'(x)$. Then as before $(f(x) - g(x))' = f'(x) - g'(x) = 0$, so $f(x) - g(x) = k$. Again this means that if we find just a single function $g(x)$ with a certain derivative, then every other function with the same derivative must be of the form $g(x) + k$.

EXAMPLE 6.5.3 Describe all functions that have derivative $5x - 3$. It's easy to find one: $g(x) = (5/2)x^2 - 3x$ has $g'(x) = 5x - 3$. The only other functions with the same derivative are therefore of the form $f(x) = (5/2)x^2 - 3x + k$.

Alternately, though not obviously, you might have first noticed that $g(x) = (5/2)x^2 - 3x + 47$ has $g'(x) = 5x - 3$. Then every other function with the same derivative must have the form $f(x) = (5/2)x^2 - 3x + 47 + k$. This looks different, but it really isn't. The functions of the form $f(x) = (5/2)x^2 - 3x + k$ are exactly the same as the ones of the form $f(x) = (5/2)x^2 - 3x + 47 + k$. For example, $(5/2)x^2 - 3x + 10$ is the same as $(5/2)x^2 - 3x + 47 + (-37)$, and the first is of the first form while the second has the second form. □

This is worth calling a theorem:

THEOREM 6.5.4 If $f'(x) = g'(x)$ for every $x \in (a, b)$, then for some constant k, $f(x) = g(x) + k$ on the interval (a, b). ■

EXAMPLE 6.5.5 Describe all functions with derivative $\sin x + e^x$. One such function is $-\cos x + e^x$, so all such functions have the form $-\cos x + e^x + k$. □

Exercises 6.5.

1. Let $f(x) = x^2$. Find a value $c \in (-1, 2)$ so that $f'(c)$ equals the slope between the endpoints of $f(x)$ on $[-1, 2]$. ⇒

2. Verify that $f(x) = x/(x + 2)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 4]$ and then find all of the values, c, that satisfy the conclusion of the theorem. ⇒

3. Verify that $f(x) = 3x/(x + 7)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-2, 6]$ and then find all of the values, c, that satisfy the conclusion of the theorem.

4. Let $f(x) = \tan x$. Show that $f(\pi) = f(2\pi) = 0$ but there is no number $c \in (\pi, 2\pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?

5. Let $f(x) = (x - 3)^{-2}$. Show that there is no value $c \in (1, 4)$ such that $f'(c) = (f(4) - f(1))/(4 - 1)$. Why is this not a contradiction of the Mean Value Theorem?

6. Describe all functions with derivative $x^2 + 47x - 5$. ⇒

7. Describe all functions with derivative $\dfrac{1}{1 + x^2}$. ⇒

8. Describe all functions with derivative $x^3 - \dfrac{1}{x}$. ⇒

9. Describe all functions with derivative $\sin(2x)$. ⇒

10. Show that the equation $6x^4 - 7x + 1 = 0$ does not have more than two distinct real roots.

11. Let f be differentiable on $\mathbb{R}$. Suppose that $f'(x) \neq 0$ for every x. Prove that f has at most one real root.

12. Prove that for all real x and y $|\cos x - \cos y| \le |x - y|$. State and prove an analogous result involving sine.

13. Show that $\sqrt{1 + x} \le 1 + (x/2)$ if $-1 < x < 1$.

A

Selected Answers

2.1.1. $-5,\ -2.47106145,\ -2.4067927,\ -2.400676,\ -2.4$

2.1.2. $-4/3,\ -24/7,\ 7/24,\ 3/4$

2.1.3. $-0.107526881,\ -0.11074197,\ -0.1110741,\ \dfrac{-1}{3(3+\Delta x)} \to \dfrac{-1}{9}$

2.1.4. $\dfrac{3+3\Delta x+\Delta x^2}{1+\Delta x} \to 3$

2.1.5. $3.31,\ 3.003001,\ 3.0000,\ 3+3\Delta x+\Delta x^2 \to 3$

2.1.6. m

2.2.1. 10, 25/2, 20, 15, 25, 35.

2.2.2. $5,\ 4.1,\ 4.01,\ 4.001,\ 4+\Delta t \to 4$

2.2.3. $-10.29,\ -9.849,\ -9.8049,\ -9.8-4.9\Delta t \to -9.8$

2.3.1. 7

2.3.2. 5

2.3.3. 0

2.3.4. undefined

2.3.5. 1/6

2.3.6. 0

2.3.7. 3

2.3.8. 172

2.3.9. 0

2.3.10. 2

2.3.11. does not exist

2.3.12. $\sqrt{2}$

2.3.13. $3a^2$

2.3.14. 512

2.3.15. -4

2.3.16. 0

2.3.18. (a) 8, (b) 6, (c) dne, (d) -2, (e) -1, (f) 8, (g) 7, (h) 6, (i) 3, (j) $-3/2$, (k) 6, (l) 2

2.4.1. $-x/\sqrt{169-x^2}$

2.4.2. $-9.8t$

2.4.3. $2x+1/x^2$

2.4.4. $2ax+b$

2.4.5. $3x^2$

2.4.8. $-2/(2x+1)^{3/2}$

2.4.9. $5/(t+2)^2$

2.4.10. $y=-13x+17$

2.4.11. -8

2.5.6. -0.5 or 1.3 or 3.2

3.1.1. $100x^{99}$

3.1.2. $-100x^{-101}$

3.1.3. $-5x^{-6}$

3.1.4. $\pi x^{\pi-1}$

3.1.5. $(3/4)x^{-1/4}$

3.1.6. $-(9/7)x^{-16/7}$

3.2.1. $15x^2+24x$

3.2.2. $-20x^4+6x+10/x^3$

3.2.3. $-30x+25$

3.2.4. $6x^2+2x-8$

3.2.5. $3x^2+6x-1$

3.2.6. $9x^2-x/\sqrt{625-x^2}$

3.2.7. $y=13x/4+5$

3.2.8. $y=24x-48-\pi^3$

3.2.9. $-49t/5+5$, $-49/5$

3.2.11. $\sum_{k=1}^{n} k a_k x^{k-1}$

3.2.12. $x^3/16-3x/4+4$

3.3.1. $3x^2(x^3-5x+10)+x^3(3x^2-5)$

3.3.2. $(x^2+5x-3)(5x^4-18x^2+6x-7)+(2x+5)(x^5-6x^3+3x^2-7x+1)$

3.3.3. $\dfrac{\sqrt{625-x^2}}{2\sqrt{x}}-\dfrac{x\sqrt{x}}{\sqrt{625-x^2}}$

3.3.4. $\dfrac{-1}{x^{19}\sqrt{625-x^2}}-\dfrac{20\sqrt{625-x^2}}{x^{21}}$

3.3.5. $f'=4(2x-3)$, $y=4x-7$

3.4.1. $\dfrac{3x^2}{x^3-5x+10}-\dfrac{x^3(3x^2-5)}{(x^3-5x+10)^2}$

3.4.2. $\dfrac{2x+5}{x^5-6x^3+3x^2-7x+1}-\dfrac{(x^2+5x-3)(5x^4-18x^2+6x-7)}{(x^5-6x^3+3x^2-7x+1)^2}$

3.4.3. $\dfrac{1}{2\sqrt{x}\sqrt{625-x^2}}+\dfrac{x^{3/2}}{(625-x^2)^{3/2}}$

3.4.4. $\dfrac{-1}{x^{19}\sqrt{625-x^2}}-\dfrac{20\sqrt{625-x^2}}{x^{21}}$

3.4.5. $y=17x/4-41/4$

3.4.6. $y=11x/16-15/16$

3.4.8. $y=19/169-5x/338$

3.4.9. $13/18$

3.5.1. $4x^3-9x^2+x+7$

3.5.2. $3x^2-4x+2/\sqrt{x}$

3.5.3. $6(x^2+1)^2x$

3.5.4. $\sqrt{169-x^2}-x^2/\sqrt{169-x^2}$

3.5.5. $(2x-4)\sqrt{25-x^2}-(x^2-4x+5)x/\sqrt{25-x^2}$

3.5.6. $-x/\sqrt{r^2-x^2}$

3.5.7. $2x^3/\sqrt{1+x^4}$

3.5.8. $\dfrac{1}{4\sqrt{x}(5-\sqrt{x})^{3/2}}$

3.5.9. $6+18x$

3.5.10. $\dfrac{2x+1}{1-x}+\dfrac{x^2+x+1}{(1-x)^2}$

3.5.11. $-1/\sqrt{25-x^2}-\sqrt{25-x^2}/x^2$

3.5.12. $\dfrac{1}{2}\left(\dfrac{-169}{x^2}-1\right)\Big/\sqrt{\dfrac{169}{x}-x}$

3.5.13. $\dfrac{3x^2-2x+1/x^2}{2\sqrt{x^3-x^2-(1/x)}}$

3.5.14. $\dfrac{300x}{(100-x^2)^{5/2}}$

3.5.15. $\dfrac{1+3x^2}{3(x+x^3)^{2/3}}$

3.5.16. $\left(4x(x^2+1)+\dfrac{4x^3+4x}{2\sqrt{1+(x^2+1)^2}}\right)\Big/ 2\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$

3.5.17. $5(x+8)^4$

3.5.18. $-3(4-x)^2$

3.5.19. $6x(x^2+5)^2$

3.5.20. $-12x(6-2x^2)^2$

3.5.21. $24x^2(1-4x^3)^{-3}$

3.5.22. $5+5/x^2$

3.5.23. $-8(4x-1)(2x^2-x+3)^{-3}$

3.5.24. $1/(x+1)^2$

3.5.25. $3(8x-2)/(4x^2-2x+1)^2$

3.5.26. $-3x^2+5x-1$

3.5.27. $6x(2x-4)^3+6(3x^2+1)(2x-4)^2$

3.5.28. $-2/(x-1)^2$

3.5.29. $4x/(x^2+1)^2$

3.5.30. $(x^2-6x+7)/(x-3)^2$

3.5.31. $-5/(3x-4)^2$

3.5.32. $60x^4+72x^3+18x^2+18x-6$

3.5.33. $(5-4x)/((2x+1)^2(x-3)^2)$

3.5.34. $1/(2(2+3x)^2)$

3.5.35. $56x^6+72x^5+110x^4+100x^3+60x^2+28x+6$

3.5.36. $y=23x/96-29/96$

3.5.37. $y=3-2x/3$

3.5.38. $y=13x/2-23/2$

3.5.39. $y=2x-11$

3.5.40. $y=\dfrac{20+2\sqrt{5}}{5\sqrt{4+\sqrt{5}}}x+\dfrac{3\sqrt{5}}{5\sqrt{4+\sqrt{5}}}$

4.1.1. $2n\pi-\pi/2$, any integer n

4.1.2. $n\pi\pm\pi/6$, any integer n

4.1.3. $(\sqrt{2}+\sqrt{6})/4$

4.1.4. $-(1+\sqrt{3})/(1-\sqrt{3})=2+\sqrt{3}$

4.1.11. $t=\pi/2$

4.3.1. 5

4.3.2. $7/2$

4.3.3. $3/4$

4.3.4. 1

4.3.5. $-\sqrt{2}/2$

4.3.6. 7

4.3.7. 2

4.4.1. $\sin(\sqrt{x})\cos(\sqrt{x})/\sqrt{x}$

4.4.2. $\dfrac{\sin x}{2\sqrt{x}}+\sqrt{x}\cos x$

4.4.3. $-\dfrac{\cos x}{\sin^2 x}$

4.4.4. $\dfrac{(2x+1)\sin x-(x^2+x)\cos x}{\sin^2 x}$

4.4.5. $\dfrac{-\sin x\cos x}{\sqrt{1-\sin^2 x}}$

4.5.1. $\cos^2 x-\sin^2 x$

4.5.2. $-\sin x\cos(\cos x)$

4.5.3. $\dfrac{\tan x+x\sec^2 x}{2\sqrt{x\tan x}}$

4.5.4. $\dfrac{\sec^2 x(1+\sin x)-\tan x\cos x}{(1+\sin x)^2}$

4.5.5. $-\csc^2 x$

4.5.6. $-\csc x\cot x$

4.5.7. $3x^2\sin(23x^2)+46x^4\cos(23x^2)$

4.5.8. 0

4.5.9. $-6\cos(\cos(6x))\sin(6x)$

4.5.10. $\dfrac{\sec\theta\tan\theta}{(1+\sec\theta)^2}=\dfrac{\sin\theta}{(\cos\theta+1)^2}$

4.5.11. $5t^4\cos(6t)-6t^5\sin(6t)$

4.5.12. $3t^2(\sin(3t)+t\cos(3t))/\cos(2t)+2t^3\sin(3t)\sin(2t)/\cos^2(2t)$

4.5.13. $n\pi/2$, any integer n

4.5.14. $\pi/2+n\pi$, any integer n

4.5.15. $y=\sqrt{3}x/2+3/4-\sqrt{3}\pi/6$

4.5.16. $y=8\sqrt{3}x+4-8\sqrt{3}\pi/3$

4.5.17. $y=3\sqrt{3}x/2-\sqrt{3}\pi/4$

4.5.18. $\pi/6+2n\pi$, $5\pi/6+2n\pi$, any integer n

4.7.1. $2\ln(3)x3^{x^2}$

4.7.2. $\dfrac{\cos x-\sin x}{e^x}$

4.7.3. $2e^{2x}$

4.7.4. $e^x\cos(e^x)$

4.7.5. $\cos(x)e^{\sin x}$

4.7.6. $x^{\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$

4.7.7. $3x^2e^x + x^3e^x$

4.7.8. $1 + 2^x \ln(2)$

4.7.9. $-2x\ln(3)(1/3)^{x^2}$

4.7.10. $e^{4x}(4x-1)/x^2$

4.7.11. $(3x^2+3)/(x^3+3x)$

4.7.12. $-\tan(x)$

4.7.13. $(1-\ln(x^2))/(x^2\sqrt{\ln(x^2)})$

4.7.14. $\sec(x)$

4.7.15. $x^{\cos(x)}(\cos(x)/x - \sin(x)\ln(x))$

4.7.20. e

4.8.1. x/y

4.8.2. $-(2x+y)/(x+2y)$

4.8.3. $(2xy-3x^2-y^2)/(2xy-3y^2-x^2)$

4.8.4. $\sin(x)\sin(y)/(\cos(x)\cos(y))$

4.8.5. $-\sqrt{y}/\sqrt{x}$

4.8.6. $(y\sec^2(x/y)-y^2)/(x\sec^2(x/y)+y^2)$

4.8.7. $(y-\cos(x+y))/(\cos(x+y)-x)$

4.8.8. $-y^2/x^2$

4.8.9. 1

4.8.12. $y = 2x \pm 6$

4.8.13. $y = x/2 \pm 3$

4.8.14. $(\sqrt{3}, 2\sqrt{3})$, $(-\sqrt{3}, -2\sqrt{3})$, $(2\sqrt{3}, \sqrt{3})$, $(-2\sqrt{3}, -\sqrt{3})$

4.8.15. $y = 7x/\sqrt{3} - 8/\sqrt{3}$

4.8.16. $y = (-y_1^{1/3}x + y_1^{1/3}x_1 + x_1^{1/3}y_1)/x_1^{1/3}$

4.8.17. $(y-y_1)/(x-x_1) = (2x_1^3 + 2x_1y_1^2 - x_1)/(2y_1^3 + 2y_1x_1^2 + y_1)$

4.9.3. $\dfrac{-1}{1+x^2}$

4.9.5. $\dfrac{2x}{\sqrt{1-x^4}}$

4.9.6. $\dfrac{e^x}{1+e^{2x}}$

4.9.7. $-3x^2\cos(x^3)/\sqrt{1-\sin^2(x^3)}$

4.9.8. $\dfrac{2}{(\arcsin x)\sqrt{1-x^2}}$

4.9.9. $-e^x/\sqrt{1-e^{2x}}$

4.9.10. 0

4.9.11. $\dfrac{(1+\ln x)x^x}{\ln 5(1+x^{2x})\arctan(x^x)}$

5.1.6. none

5.1.7. none

5.1.8. min at $x = 7\pi/12 + k\pi$, max at $x = -\pi/12 + k\pi$, for integer k.

5.1.9. none

5.1.10. local max at $x = 5$

5.1.11. local min at $x = 49$

5.1.12. local min at $x = 0$

5.1.15. one

5.2.1. min at $x = 1/2$

5.2.2. min at $x = -1$, max at $x = 1$

5.2.3. max at $x = 2$, min at $x = 4$

5.2.4. min at $x = \pm 1$, max at $x = 0$.

5.2.5. min at $x = 1$

5.2.6. none

5.2.7. none

5.2.8. min at $x = 7\pi/12 + k\pi$, max at $x = -\pi/12 + k\pi$, for integer k.

5.2.9. none

5.2.10. max at $x = 0$, min at $x = \pm 11$

5.2.11. min at $x = -3/2$, neither at $x = 0$

5.2.13. min at $n\pi$, max at $\pi/2 + n\pi$

5.2.14. min at $2n\pi$, max at $(2n + 1)\pi$

5.2.15. min at $\pi/2+2n\pi$, max at $3\pi/2+2n\pi$

5.3.1. min at $x = 1/2$

5.3.2. min at $x = -1$, max at $x = 1$

5.3.3. max at $x = 2$, min at $x = 4$

5.3.4. min at $x = \pm 1$, max at $x = 0$.

5.3.5. min at $x = 1$

5.3.6. none

5.3.7. none

5.3.8. min at $x = 7\pi/12 + n\pi$, max at $x = -\pi/12 + n\pi$, for integer n.

5.3.9. max at $x = 63/64$

5.3.10. max at $x = 7$

5.3.11. max at $-5^{-1/4}$, min at $5^{-1/4}$

5.3.12. none

5.3.13. max at -1, min at 1

5.3.14. min at $2^{-1/3}$

5.3.15. none

5.3.16. min at $n\pi$

5.3.17. max at $n\pi$, min at $\pi/2 + n\pi$

5.3.18. max at $\pi/2+2n\pi$, min at $3\pi/2+2n\pi$

5.4.1. concave up everywhere

5.4.2. concave up when $x < 0$, concave down when $x > 0$

5.4.3. concave down when $x < 3$, concave up when $x > 3$

5.4.4. concave up when $x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$, concave down when $-1/\sqrt{3} < x < 1/\sqrt{3}$

5.4.5. concave up when $x < 0$ or $x > 2/3$, concave down when $0 < x < 2/3$

5.4.6. concave up when $x < 0$, concave down when $x > 0$

5.4.7. concave up when $x < -1$ or $x > 1$, concave down when $-1 < x < 0$ or $0 < x < 1$

5.4.8. concave down on $((8n-1)\pi/4, (8n+3)\pi/4)$, concave up on $((8n+3)\pi/4, (8n+7)\pi/4)$, for integer n

5.4.9. concave down everywhere

5.4.10. concave up on $(-\infty, (21-\sqrt{497})/4)$ and $(21+\sqrt{497})/4, \infty)$

5.4.11. concave up on $(0, \infty)$

5.4.12. concave down on $(2n\pi/3, (2n+1)\pi/3)$

5.4.13. concave up on $(0, \infty)$

5.4.14. concave up on $(-\infty, -1)$ and $(0, \infty)$

5.4.15. concave down everywhere

5.4.16. concave up everywhere

5.4.17. concave up on $(\pi/4+n\pi, 3\pi/4+n\pi)$

5.4.18. inflection points at $n\pi$, $\pm\arcsin(\sqrt{2/3}) + n\pi$

5.4.19. up/incr: $(3, \infty)$, up/decr: $(-\infty, 0)$, $(2, 3)$, down/decr: $(0, 2)$

6.1.1. max at $(2, 5)$, min at $(0, 1)$

6.1.2. 25×25

6.1.3. $P/4 \times P/4$

6.1.4. $w = l = 2 \cdot 5^{2/3}$, $h = 5^{2/3}$, $h/w = 1/2$

6.1.5. $\sqrt[3]{100} \times \sqrt[3]{100} \times 2\sqrt[3]{100}$, $h/s = 2$

6.1.6. $w = l = 2^{1/3}V^{1/3}$, $h = V^{1/3}/2^{2/3}$, $h/w = 1/2$

6.1.7. 1250 square feet

6.1.8. $l^2/8$ square feet

6.1.9. \$5000

6.1.10. 100

6.1.11. r^2

6.1.12. $h/r = 2$

6.1.13. $h/r = 2$

6.1.14. $r = 5$ cm, $h = 40/\pi$ cm, $h/r = 8/\pi$

6.1.15. $8/\pi$

6.1.16. $4/27$

6.1.17. Go direct from A to D.

6.1.18. (a) 2, (b) 7/2

6.1.19. $\dfrac{\sqrt{3}}{6} \times \dfrac{\sqrt{3}}{6} + \dfrac{1}{2} \times \dfrac{1}{4} - \dfrac{\sqrt{3}}{12}$

6.1.20. (a) $a/6$, (b) $(a+b-\sqrt{a^2-ab+b^2})/6$

6.1.21. 1.5 meters wide by 1.25 meters tall

6.1.22. If $k \le 2/\pi$ the ratio is $(2-k\pi)/4$; if $k \ge 2/\pi$, the ratio is zero: the window should be semicircular with no rectangular part.

6.1.23. a/b

6.1.24. $w = 2r/\sqrt{3}$, $h = 2\sqrt{2}r/\sqrt{3}$

6.1.25. $1/\sqrt{3} \approx 58\%$

6.1.26. $18 \times 18 \times 36$

6.1.27. $r = 5/(2\pi)^{1/3} \approx 2.7$ cm, $h = 5 \cdot 2^{5/3}/\pi^{1/3} = 4r \approx 10.8$ cm

6.1.28. $h = \dfrac{750}{\pi}\left(\dfrac{2\pi^2}{750^2}\right)^{1/3}$, $r = \left(\dfrac{750^2}{2\pi^2}\right)^{1/6}$

6.1.29. $h/r = \sqrt{2}$

6.1.30. The ratio of the volume of the sphere to the volume of the cone is $1033/4096 + 33/4096\sqrt{17} \approx 0.2854$, so the cone occupies approximately 28.54% of the sphere.

6.1.31. P should be at distance $c\sqrt[3]{a}/(\sqrt[3]{a}+\sqrt[3]{b})$ from charge A.

6.1.32. 1/2

6.1.33. \$7000

6.1.34. There is a critical point when $\sin\theta_1/v_1 = \sin\theta_2/v_2$, and the second derivative is positive, so there is a minimum at the critical point.

6.2.1. $1/(16\pi)$ cm/s

6.2.2. $3/(1000\pi)$ meters/second

6.2.3. 1/4 m/s

6.2.4. $-6/25$ m/s

6.2.5. 80π mi/min

6.2.6. $3\sqrt{5}$ ft/s

6.2.7. $20/(3\pi)$ cm/s

6.2.8. 13/20 ft/s

6.2.9. $5\sqrt{10}/2$ m/s

6.2.10. 75/64 m/min

6.2.11. $145\pi/72$ m/s

6.2.12. $25\pi/144$ m/min

6.2.13. $\pi\sqrt{2}/36$ ft^3/s

6.2.14. tip: 6 ft/s, length: 5/2 ft/s

6.2.15. tip: 20/11 m/s, length: 9/11 m/s

6.2.16. $380/\sqrt{3} - 150 \approx 69.4$ mph

6.2.17. $500/\sqrt{3} - 200 \approx 88.7$ km/hr

6.2.18. 18 m/s

6.2.19. $136\sqrt{475}/19 \approx 156$ km/hr

6.2.20. -50 m/s

6.2.21. 68 m/s

6.2.22. $3800/\sqrt{329} \approx 210$ km/hr

6.2.23. $820/\sqrt{329} + 150\sqrt{57}/\sqrt{47} \approx 210$ km/hr

6.2.24. 4000/49 m/s

6.2.25. (a) $x = a\cos\theta - a\sin\theta\cot(\theta+\beta) = a\sin\beta/\sin(\theta+\beta)$, (c) $\dot{x} \approx 3.79$ cm/s

6.3.1. $x_3 = 1.475773162$

6.3.2. 2.15

6.3.3. 3.36

6.3.4. 2.19 or 1.26

6.4.1. $\Delta y = 65/16$, $dy = 2$

6.4.2. $\Delta y = \sqrt{11/10} - 1$, $dy = 0.05$

6.4.3. $\Delta y = \sin(\pi/50)$, $dy = \pi/50$

6.4.4. $dV = 8\pi/25$

6.5.1. $c = 1/2$

6.5.2. $c = \sqrt{18} - 2$

6.5.6. $x^3/3 + 47x^2/2 - 5x + k$

6.5.7. $\arctan x + k$

6.5.8. $x^4/4 - \ln x + k$

6.5.9. $-\cos(2x)/2 + k$

B

Useful Formulas

Algebra

Remember that the common algebraic operations have **precedences** relative to each other: for example, multiplication and division take precedence over addition and subtraction, but are "tied" with each other. In the case of ties, work left to right. This means, for example, that $1/2x$ means $(1/2)x$: do the division, then the multiplication in left to right order. It sometimes is a good idea to use more parentheses than strictly necessary, for clarity, but it is also a bad idea to use too many parentheses.

Completing the square: $x^2 + bx + c = (x + \frac{b}{2})^2 - \frac{b^2}{4} + c$.

Quadratic formula: the roots of $ax^2 + bx + c$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Exponent rules:

$$\begin{aligned} a^b \cdot a^c &= a^{b+c} \\ \frac{a^b}{a^c} &= a^{b-c} \\ (a^b)^c &= a^{bc} \\ a^{1/b} &= \sqrt[b]{a} \end{aligned}$$

Geometry

Circle: circumference $= 2\pi r$, area $= \pi r^2$.

Ellipse: area $= \pi ab$, where $2a$ and $2b$ are the lengths of the axes of the ellipse.

Sphere: vol $= 4\pi r^3/3$, surface area $= 4\pi r^2$.

Cylinder: vol $= \pi r^2 h$, lateral area $= 2\pi rh$, total surface area $= 2\pi rh + 2\pi r^2$.

Cone: vol $= \pi r^2 h/3$, lateral area $= \pi r\sqrt{r^2+h^2}$, total surface area $= \pi r\sqrt{r^2+h^2} + \pi r^2$.

Analytic geometry

Point-slope formula for straight line through the point (x_0, y_0) with slope m: $y = y_0 + m(x - x_0)$.

Circle with radius r centered at (h, k): $(x-h)^2 + (y-k)^2 = r^2$.

Ellipse with axes on the x-axis and y-axis: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Trigonometry

$\sin(\theta) = \text{opposite}/\text{hypotenuse}$

$\cos(\theta) = \text{adjacent}/\text{hypotenuse}$

$\tan(\theta) = \text{opposite}/\text{adjacent}$

$\sec(\theta) = 1/\cos(\theta)$

$\csc(\theta) = 1/\sin(\theta)$

$\cot(\theta) = 1/\tan(\theta)$

$\tan(\theta) = \sin(\theta)/\cos(\theta)$

$\cot(\theta) = \cos(\theta)/\sin(\theta)$

$\cos^2(\theta) + \sin^2(\theta) = 1$
$\tan^2(\theta) + 1 = \sec^2(\theta)$
$\sec^2(\theta) - 1 = \tan^2(\theta)$
$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$

$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$

$\sin(\theta + \pi) = -\sin(\theta)$

$\cos(\theta + \pi) = -\cos(\theta)$

Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos A$

Law of sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Sine of sum of angles: $\sin(x+y) = \sin x\cos y + \cos x\sin y$

Sine of double angle: $\sin(2x) = 2\sin x\cos x$

Sine of difference of angles: $\sin(x - y) = \sin x\cos y - \cos x\sin y$

Cosine of sum of angles: $\cos(x + y) = \cos x\cos y - \sin x\sin y$

Cosine of double angle: $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Cosine of difference of angles: $\cos(x - y) = \cos x\cos y + \sin x\sin y$

Tangent of sum of angles: $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

$\sin^2(\theta)$ and $\cos^2(\theta)$ formulas:

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
\tan^2(\theta) + 1 &= \sec^2(\theta) \\
\sin^2(\theta) &= \frac{1 - \cos(2\theta)}{2} \\
\cos^2(\theta) &= \frac{1 + \cos(2\theta)}{2}
\end{aligned}$$

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